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FUNCTIONALITY IN MATHEMATICAL INSTRUCTION IN SCHOOLS AND COLLEGES

By E. R. HEDRICK

(Address delivered before the Mathematical Association of America. Toronto, December 30, 1921.)

1. *General Ideas.* The topic on which I am to speak to you was emphasized by the National Committee on Mathematical Requirements of this Association in its first report entitled "The Reorganization of the First Courses in Secondary School Mathematics," which was published as Secondary School Circular No. 5 by the U. S. Bureau of Education. In that report, the Committee says:

> General Ideas.—The one great idea which is sufficient in scope to unify the course is that of the functional relation. The concept of a variable and of the dependence of one variable upon another is of fundamental importance for everyone. It is true that the general and abstract form of these concepts can become significant to the pupil only as a result of very considerable mathematical experience and training. There is nothing in either concept, however, which prevents the presentation of specific concrete examples and illustrations of dependence even in the early parts of the course. Means to this end will be found in connection with the tabulation of data, and the study of the formula and the graph and of their uses.
>
> The primary and underlying principle of the course, particularly in connection with algebra and trigonometry, should be the notion of relationship between variables, including the methods of determining and expressing such relationship. The notion of relationship is fundamental both in algebra and in geometry. The teacher should have it constantly in mind, and the pupil's advancement should be consciously directed along the lines which will present first one and then another of the details upon which finally the formation of the general concept of functionality depends.

Although this statement contains in brief the substance of the whole matter, it was felt that some elaboration of the ideas was necessary to convey adequately to teachers its full import. Accordingly, the Committee requested me to prepare a more detailed statement; and this statement, after some revision, was adopted by the Committee, and was issued under the title "The Function Concept in Secondary School Mathematics" by the U. S. Bureau of Education as Secondary School Circular No. 8, June, 1921. I am assuming that many, if not all of you, have seen these two reports, and I shall not repeat them unduly.

These, as well as other reports of the Committee, may be obtained through the Chairman of the Committee or directly from Washington; they will be contained also in the full report of the Committee, which is soon to be issued in final form.

My address today would have been incomplete without at least some mention of these reports. On the other hand, these reports refer almost exclusively to the early work of the first two years of Secondary Schools. The principles announced in them are of far wider application, however, and I desire to present to you today what I regard as the very vital application of these principles not alone for mathematics, but even more for education as a whole; and not alone in secondary schools, but even more in colleges and universities. I shall point out in detail the opportunity and the need for emphasis on the functional elements in mathematical teaching in schools of every grade; and I shall try to show that this type of mathematics is of direct value, both to the individual for his selfish or personal good, and also to society and to the state, for the advancement of scientific knowledge, for the social and economic problems of the community that involve quantitative relations, and for the development of more intelligent citizenship. These phases of mathematical instruction have been too long ignored not only by professors of education who may be blamelessly ignorant of their very existence, but also by teachers and professors of mathematics, in whom ignorance is inexcusable. To continue to ignore what is probably the greatest asset of mathematics as an element in public education is not merely a demonstration of inefficiency; it is a crime against mathematics and against the cause of education in general.

2. *The Function Idea.* To such an audience as this it is unnecessary to explain what is meant by the function idea. Indeed, the danger would seem to me to be that your great familiarity with this notion may breed contempt of its elementary phases, and contempt of those who innocently know nothing about it. Denuded of our traditional phraseology, the idea of functionality is nothing else than the notion of relationships between quantities, and the manner in which changes in one of two related quantities produces—or is accompanied by—changes in the other. To you it will seem that the existence of such relationships is so self-evident, and that the effects of changes in one

quantity upon another are so easy to trace, that we might assume such ideas as part of the mental equipment of all human beings. This is far from true. The acquisition of such ideas is a very slow process. It must be begun early in a very simple manner; it must be presented first only by individual instances of a simple and numerical character; it must be fixed in the mind by repetition after repetition, and by instance after instance, until thinking in such terms becomes habitual with the individual. Only in this way can an individual acquire what has been called the "habit of functional thinking."

That relationships between quantities, and the effects upon one quantity of changes in a related quantity, enter into all human thinking about quantities, including a great part of the daily activities of ordinary life, will be admitted by all. The danger is that those already competent in such thinking will not recognize any need for training in it; and in particular that teachers of mathematics will overlook this obvious element in mathematical teaching because it seems to them self-explanatory. There will be many who think habitually about quantitative relations who will be unconscious of the mathematical foundation for the very thinking that they do.

The point I am striving to make is most clearly illustrated by a comparison with the ordinary use of English in speaking and in writing. Beneath the conscious habit of swift use of correct English there lies a subconscious knowledge of grammatical form acquired either by conscious study or by the more painful path of rough experience. Without this substratum of grammatical form, correct use of English is not only difficult—it is unthinkable. To acquire a correct and ready command of English, one may follow either the ready path of conscious education, or the thornier one of experience by trial and error. However acquired, the goal toward which we strive is the submergence into subconsciousness of the slow rules of grammar, and the formation of habits which alone render possible that swiftness of application which is indispensable in actual speech.

Similarly, thinking about quantities cannot be conducted with the necessary facility and speed until all conscious thought of algebraic rules has sunk into the subconscious portions of our minds, and until correct habits of thought concerning relationships between quantities have become so ingrained that we are

totally unconscious of the use of algebra, just as we are totally unconscious of the use of grammar when we speak. To assert that correct thinking about quantitative relationships is impossible without a previous study of algebra is just as false as to assert that correct speech is impossible without a previous conscious study of grammar. To assert that any course in algebra—for example, a course filled with formal definitions and practice in the manipulation of symbols to the exclusion of thought processes about quantitative relationships—to assert that any such course will furnish the requisite background for correct thinking about quantitative relationships that occur in actual life, is just as foolish as to suppose that any course in grammar—for example, a course wholly given over to formal definitions of parts of speech, and to declensions and conjugations—would furnish an adequate background for correct usage of English.

It is not definitions that we need. For my part, I would as leave have a student utterly unaware of the existence of the word "function." What we do want is that the student shall be vividly aware of the notion itself, of the existence of actual functions, of the meaning of a variety of relationships between quantities whose meaning is clear to him, and of the manner in which changes in one quantity affect the values of another, associated, quantity. This can be attained if we attempt it; it cannot be attained by keeping our courses solely to formalities of definitions and manipulations of symbols; it cannot be attained by the drawing of a few graphs in a perfunctory manner; it cannot be attained by devoting three recitations—or three months—to learning set definitions; it cannot be learned by any means readily tested by formal examinations; it can be learned if we teachers of mathematics solemnly resolve to call attention to every case in which quantities are related, and if we require attention and thought upon the manner in which changes in one quantity are associated with changes in another.

It may be asked how and when opportunity for such discussions will arise. The answer is that such an opportunity presents itself whenever we are dealing with quantities that are in any way related to other quantities. Lack of such opportunity means either that we are not dealing with quantities, or that we are dealing with quantities only one at a time, and that those we use are not related to or dependent upon any others.

Now quantities that have any meaning usually are related to other quantities, and their values are affected by the values of those others. How may we manage, indeed, to deal with quantities at all without dealing with related quantities? The only way to escape relationships between quantities is to deal with quantities so abstract that the relationships are lost to sight. Does it not occur to you that such abstraction is rather common in our traditional courses?

Functional relations—that is relations between quantities—will occur on every page of every book on mathematics unless we suppress them. We have been suppressing them. It is our business to emphasize them, not during one week nor one month, nor one year, but precisely whenever they occur. Since mathematics deals almost exclusively with relations between quantities, you may be fairly sure that it is something other than mathematics which is being emphasized if such relationships do not appear. And this is true. In algebra, the shorthand has supplanted the real mathematics to an alarming degree. In geometry, formal logic has all but banished mathematical thinking. Shorthand in algebra is extremely desirable—in order to express relationships between quantities; but to spend all the time on the shorthand—the symbolism—without any realization of the relationships for which the shorthand was to be the handmaiden, is to spread a feast without any guests. Logic is our supreme instrument for the discovery and confirmation of geometric and other relationships; but I have seen courses in geometry so crammed with formal logic that geometric relationships utterly failed to appear, just as a child may gorge itself with food that would furnish needed energy until all action for which the food would be good is utterly estopped.

It is such courses in algebra and in geometry that have furnished just cause for educational criticism. Algebra that has turned into shorthand, and shorthand that is never used. Geometry that is turned into formal logic, to the abandonment of geometric relationships. Such algebra, such geometry, these are no longer branches of mathematics if quantitative relations are not studied.

3. *Review of Algebra.* The criterion as to whether a given piece of work is algebraic is not the number of symbols it contains. It may or it may not contain symbols. The real criterion

is whether there is in it any question of relationship between numerical quantities. The false standard by which algebra is often judged, by the signs and symbols and the detached letters, is a criterion of shorthand only. Algebra can be written without shorthand. Shorthand can be written without algebraic content. A course in algebra should be a course in relationships between numerical quantities, with as much shorthand as may be convenient. The courses criticized are often little more than courses in shorthand, with as little algebra—*i. e.*, as little quantitative relationship, as can be managed.

In the report on the function concept mentioned above, I have tried to bring out in detail the variety of quantitative relationships that exist in our courses in algebra, if only we do not suppress them. Graphs, to be sure; but graphs of quantitative situations within the grasp of the child, not graphs of formalized equations. I have not emphasized graphs at all, for the danger is that the hasty teacher may see nothing else. Every instance of the use of letters for quantities is only to express a relationship; otherwise it is really quite as useless as some educators believe it to be. Why should we write for simple interest $i = r \cdot p \cdot t$ unless we are at least to contemplate the relationship, and to note how the interest is affected by changes in the rate, principal, and time? More complex forms of relationships occur later in algebra, through formulas of science introduced as applications, and through formulas of purely mathematical origin. Thus I have emphasized such formulas as the laws of falling bodies, the law for compound interest, the law for the square of a sum. In all such cases, real appreciation is probably lacking if or until the relationships between the quantities involved is clearly thought out and understood. The contrast between the total interest at simple interest and that at compound interest is a striking case. I have detailed all this in my report more fully than I can do here.

May I emphasize the growing importance of tables of numerical values? Any such table, in the nature of things, tells many values of one of two related quantities when the other is given. It is strictly functional in character. The need for interpolation in such a table also forces functional thinking of a high order, if it is intelligently done.

Finally, the very existence of what we call practical problems depends upon the relationships between the quantities mentioned in the problem. If only one concept were mentioned, the problem must be extremely artificial or extremely simple. The study of the relations between the quantities mentioned in the problem will ordinarily lead to a solution. Failure to study these relations is the most usual source of trouble in such problems.

4. *Review of Geometry.* The case in geometry is no less striking. Here the fundamentals of the subject are the relationships that exist between geometric quantities, as opposed to the relationships between numerical quantities which constitute algebra. The relations that exist between the parts of a triangle have been developed most completely, and we recognize their functional character when we use the so-called trigonometric functions. But the parts of all geometric figures are inter-related. Geometry should make these relations stand forth. I well remember that my own first course in geometry did nothing of the sort. To watch the shape of a parallelogram change as the angles change while the sides remain fixed; to think out the resulting changes in the lengths of the diagonals; all this is real geometry. To consider the arcs that measure an angle as the vertex of the angle moves from the center of the circle out toward the circumference and then beyond the circumference, is to acquire real insight into these geometric forms.

I do not decry in the slightest degree the necessity of formal logic in geometry. It is necessary. But it is not the subject-matter of the geometry. The subject-matter consists in those relationships between geometric forms which the logic serves to demonstrate.

Changes produced in figures by the variation of one or more of its parts are of the utmost consequence not only toward a real mastery of geometry, but also for the applications. Thus a knowledge of which parts of a framework are necessary to make it rigid, and which must break if it is to change form, are of the utmost consequence in all building operations. Ability to think such geometric relationships quickly and accurately is a part of the reasonable mental equipment of all grown men and of most women. I do not think that it is taught adequately in our courses in geometry.

Motion, and the properties of moving figures, occur in the lives of all people. To think accurately about machinery, about plans, about patterns, about motions, geometric habits of functional thinking are quite necessary. A study of geometry, to be thorough and to make for efficiency in life, demands habits of functional thinking regarding geometric forms. Without it, geometry sinks to a level of isolated figures, and is often little more than a quickly forgotten exercise in logistics.

5. *Algebra for Girls. Other Special Classes.* A question that I have often heard asked, and one that still amazes me, is concerning the usefulness of algebra for girls. Perhaps this question is inspired by that purely formalistic view of algebra that takes no account of the formation of the habit of functional thinking, *i. e.*, the habit of thinking about relations between quantities. For every woman who does anything, whether it be a man-like task outside the home, or those tasks of housewifery that are often so misprized, deals with quantities and with relations between quantities every day of every week. The question as to whether twice as much gas burned under a kettle will make potatoes cook twice as fast is of the same order of difficulty as the question as to whether compound interest on money for twice as long is twice as much. The question as to the comparative contents of two cans of tomatoes of the same shape is actually the same question as the question as to the relative volumes of similar cylinders. If one is twice as high as the other, will it hold just twice as much? Will twice as much ice make the icebox twice as cold? What is meant by twice as cold? The reading of a gasmeter is a functional question of no mean order. Will twice as much sugar make a thing twice as sweet? Will doubling the amount of soap wash the clothes twice as clean? Will doubling the amount of yeast make the loaves twice as large, or raise the bread twice as fast? What is the effect on cake of increasing the amount of butter? Are oranges twice as thick through twice as heavy? If the skins are twice as thick on the large oranges, did we miscalculate the relative values of the oranges? Does it pay to buy big potatoes as compared with small ones? Will a mail order package twice as heavy cost twice as much postage? If sent by express, would the expressage be doubled? When does it become advisable to ship by freight? Will a kitchen window twice as large let in twice as much light?

Will twice as many guests double the expense of a party: hence is it just as well to have the same number of guests at two different times as to have all at one time? Will twice the amount of food at each of half as many meals sustain life as well? Does one feed a man who works more than one who does not? How much more? If he works twice as long, should he have twice as much food? And there are those finer questions of balancing rations for children, for invalids, for old people; questions of calories and proteins and carbohydrates for the real expert; questions of bacterial growth in diseases, and in sour milk, and—sometimes—in beverages; questions of food for a mother and food for a child, content of butter fat and protein in milk and in skimmed milk, acids, sugars, starches. Quantities—quantities—relations between quantities.

Perhaps many a woman hesitates to comprehend or even to study such questions. "There are too many quantities." "I can never figure things out." "I have no head for figures." "That is too complicated for me." But all these things are well within the range of experience of most women. Why are they too hard? Why cannot women understand and master such things? Is it perhaps that they have not formed habits of quantitative thinking? Is it perhaps that some all-too-wise person has told them that algebra is of no use to girls? Is it perhaps that the teacher of algebra was too busy with the shorthand of algebraic symbols to spend time over mere questioning about relationships between quantities?

Some girls need no experience in dealing with quantities, and no habit of accurate thinking about quantitative relations. Some girls will never have to do with quantities or with relations between quantities. Who are these girls? They are those who toil not, neither do they spin. The idle have no need for accurate habits of thought. The parasite on society needs no knowledge of relationships between quantities. But all other women meet with quantities and with relationships between them. Training for parasitism—and there is woefully much of just that—need include nothing about quantities. Training for usefulness—which is doubtless far drearier—should not neglect training in habits of quantitative thinking, training of such character that finally accurate thinking is possible speedily and without consciousness of the sub-conscious basis for that accuracy, just as

speech should be accurate and rapid without thought of the sub-conscious basis for language. Those who plan curricula for girls should not overlook these things. Nor should those who teach mathematics to girls. More habits of quantitative thinking—and less shorthand. They will have need for all the knowledge they can get about quantitative relationships; they will have little need for manipulation of symbols and for intricate shorthand.

6. *College Mathematics.* Boys, too, have occasion to deal with quantities and to think accurately about relationships between quantities. I suppose that most people think that men deal with quantities more than women do. It was for that reason that I discussed the quantities women meet with somewhat in detail. If men meet more, they surely meet enough.

I have heard men say that they could never comprehend house plans, or plans for such complex structures as modern sewage systems. I know that many men cannot comprehend life insurance; hence they trust in that broken reed—the life insurance agent—blind leading the blind. We have been told rather frequently by those who have axes to grind that we—the common people—cannot understand such complex questions as railroad rates; and a few have asserted that only those who hold "a divine right" can comprehend, administer, and reap what profit they will, from these and other public corporations. Here is a very welter of quantities: compound interest, insurance, probabilities, depreciation.

I am not now speaking of such technically trained men as engineers, chemists, and other specialists. These are perfectly aware of their needs, and they are going to find the opportunity of meeting them, even when departments of mathematics are not framing their courses so as to meet these needs. They will be served. What we fail to give they will—and they should—seek elsewhere; and they will find . . . Remember descriptive geometry—a beautiful topic now all but lost to mathematics.

I choose to speak rather of that broader class who are being trained in colleges to go out into the world in dozens of different callings not so specifically technical as is engineering. The business man, in any of a hundred businesses, more especially the banker, but also good farmers, good lawyers, good storekeepers, good contractors, good manufacturers, good public service corporation men, good insurance men, all will find in their private

interests not less but more problems of a quantitative nature, more relationships between quantities than those I have mentioned in detail for the housewife. While any one of them would probably be amazed at the variety of quantitative situations which the housewife has to meet, I doubt not that any one of them would stoutly contend that his own problems are still more varied and complex. Not to tarry over the pettier ones, let me mention one quite complex problem which is met by every class just mentioned: the problem of depreciation. On the farm, if it is a good farm, tools deteriorate, and they should be valued; in a factory, machinery deteriorates, buildings deteriorate, men deteriorate; in a public service corporation, all installation deteriorates, all is subject to public valuation; the part of the banker, the lawyer, the legislator, in all of this is evident upon very slight examination. We have met such problems very badly. Better business, higher efficiency, better service to the public, and better returns to the investor, all wait upon more general comprehension of the reality of these problems and of the possibility of solving them. Waste and inefficiency, lavish buying and prodigal carelessness of the future, these go hand in hand with guessing at quantities and at the effects of some on others. Careful and true thinking on these quantitative matters will save business from its precarious position, will save untold millions of public money and of private graft, will reassure a people now harassed and worried by private and public questions that they profess not to understand because of their quantitative complexities.

I say these things. Yet do our collegiate courses tend to train our students to meet all such quantitative situations when they do meet them? To lead the less fortunate on private and public affairs that involve quantities? I doubt it. Nor is the remedy simply in the substitution of a course in the mathematics of business for a course in trigonometry. In my opinion, both are too special in their scope and too narrowly technical and manipulative in their details. We should not seek to train the classes that face us wholly for surveying, nor wholly for insurance; nor wholly for accounting. It seems to me that we should, at least in a general course not designed for engineers nor for schools of business, give the great body of students as broad an idea of the variety of situations with which mathematics deals

as we can possibly give. For them, the tangent law of trigonometry is as futile as a chapter from Bowditch's *Navigation*. But the familiarity with simple ideas such as the possibility of indirect measurement associated with right triangle solutions and tables of values, will be as likely to inspire him and to affect his life as would, for example, a knowledge of the Magna Charta, or a Description of the Glacial Period, or the Mendelian Law. Some knowledge of insurance there should be, but not a technical training in it outside of schools of business. To see that a net premium can be computed may really awaken intellectual interests undreamed of, and may seriously affect a career; at least, it is comparable with a knowledge of the Structure of the Inner Ear, or the Neanderthal Man, or Grimm's Law. But to carry the detail too far—say to actual computations with commutation columns—is to exclude the possibility of enrichment of the course by other ideas equally inviting. Thus probability, as applied in statistics, or in animal breeding, is of just as general importance and interest. But I am not intending to outline a course, or to favor any existing treatment: frankly I do not know a book that does all these thhings and also all the other things that I would class with them.

We should teach graphical processes, of course, to train for problems of life that involve quantitative relations. We should treat also the equations that are associated with these graphs; *i. e.*, we should teach analytic geometry. But again, we should have an eye to limitation on detail and perhaps another eye to the forceful presentation of the functional character of what we do present. Thus I gravely doubt whether the ordinary college man not intending to specialize in mathematics or engineering will profit either now or later from a treatment of the Normal Form of the Straight Line Equation. But I know that any one who is to meet and wrestle with quantitative relationships will profit decidedly from an appreciation of the nature of linear relationships, and the distinction between these and non--linear ones, even if he gets no further than to realize that non-linear relations are apt to occur. Apparently, many scientists and engineers have that yet to grasp effectively. On the other hand, are even our most detailed courses on analytics, as conducted at present, giving our students a real appreciation of linear relationships as a thing existent outside of books and equations? I

seriously doubt it. We have been so busy with the equations of perpendicular bisectors and of pencils of lines, that we have not found time to dwell on the intrinsic meaning, and to enforce a feeling for actual quantities that may (or may not) obey laws precisely like the equations we so unmercifully manipulate. For the mass of students, I contend that it is far more important that they comprehend that a good gas meter shows readings that are in linear relation to the amount of gas consumed, and conversely that any such meter is a good meter, than it is to know how to find the distance from a given point to a given line by analytic methods.

Time will not permit more illustration. I am not especially interested in gas meters; but you may reconstruct the entire skeleton, as Agassiz could that of a prehistoric fish, from this one rib of my mental anatomy. Another would be a like comparison between the hyperbolic character of such inverse variation as that between the pressure and the volume of a gas, as contrasted with the manipulative process of rotating that same hyperbola through forty-five degrees to reduce it to a traditional form that is more complicated!

I should at least mention the calculus, though I realize that there are limits to your patience. The calculus is par excellence the study of the manner in which functions vary. It deals primarily with the rates of such changes and with the determination of quantities whose rates of change are given. My insistence would be mainly that the student should realize both this fact itself, and the meaning of it. Too often we submerge it in a maze of technique of mere manipulation of algebraic and trigonometric symbols. True command of the calculus is not told off in the number of forms one can differentiate and integrate; it is determined by the student's grasp of the reality about the rates of change of functions, and their realization that both the differential and the integral processes occur in all forms of quantitative relationships. If they do not know this to the extent of instantly thinking of it whenever relationships between quantities present themselves, then their knowledge of the calculus is imperfect, even if they have memorized, as I did when young, all the reduction formulas for binomials differentials and for trigonometric forms. I have forgotten them, but my knowledge of the calculus is on a surer foundation. What shall

it profit a student though he know all the book, but have lost its soul? Aye, and it may profit him, for he may pass the course!

7. *Functional Character of Science and of Modern Life.* I have tried to point out how the idea of functional relationships pervades all of mathematics, or should pervade it, from the earliest courses in secondary schools to and beyond the calculus. Much of this has been lost in our traditional courses. Many of our students have gone from us unstirred by the vision of the relationships that exists in their own lives and in the world about them. Many of our teachers seem strangely negligent of this whole matter. Most of the critics of mathematics as a school subject seem totally unaware of its existence. Reform, and reform in this direction, is a paramount necessity for our courses if they are to survive as great school courses. And let me add that I shall raise no hand in defence of those courses that continue to neglect this central theme of all mathematics.

Functional relationships are not only the central theme of real mathematics; they constitute also its mightiest application in science and in the life of the modern world. When any modern science reaches the point in its groping for truth where measurement is possible, as most of them now have, almost the first serious question is of the relationships between associated quantities. First graphically, and then much later, when empiricism has exhausted its powers, also analytically, we seek in modern science to express as best we may the observed facts of nature, in the form of relationships between quantities. These are the laws of science. Is there a quantitative science in which this is not true? The outstanding problems of modern science, in things quantitative, are problems of functional dependence, in its purest form.

The problems of modern life are perhaps less strongly associated with functional dependence; but the associations are far more real and far more definite than has been expressed by any one. I have ventured to mention instances: life insurance, depreciation, compound interest on money. But I may add that wherever science touches modern life in quantitative aspects, the process is bound to be functional in character. For in what else would the business man be interested if not in the relationships between the quantities of the things of which the

scientist is to tell him? Modern life is complex through our very knowledge of many quantitative relations. From the woman who deals with balanced rations to the man who builds a bridge, or the man who values a public service corporation, accurate habits of functional thinking are now demanded and secured. I will admit, perhaps, that we have partly failed in our mission to educate our students toward easy control of such things; but I will not admit that we cannot so educate them, nor that these problems do not exist in great number and in endless variey in the modern world.

Thus far I have spoken chiefly of what I may call the selfish or personal benefits that follow from good habits of functional thinking. I have mentioned, however, many whose broader import is of public character. Thus the control of public service corporations and of life insurance companies will either be viciously bad or it will be based upon good functional thinking, for valuation of public utilities and computation of just insurance are both fundamentally mathematical and functional in character. There are other public questions of similar nature. Control of railroad and express rates are now public rather than private questions. Taxation is emphatically public, and it is highly functional in character when it touches incomes and excess profits and unearned increments. Bond issues and redemptions, amortization, the tariff, pensions, representation, and many other phases of the public business are decidedly mathematical and decidedly functional in character.

Nor will it do to say that these questions may be solved by a few experts. Fortunately—or will you say unfortunately?—we live in democracies. In an autocracy, one outstanding advantage —or is it an advantage?—is that all such public questions may be decided by the few without consultation of the many. I need not argue to you of the abuses to which that leads. Democracy is indeed worth fighting for, and we have just won a mighty fight for it against autocracy. But democracy is postulated upon widespread understanding of public questions—upon general education of the masses as well as of the classes in all that affects the nation. Legislation must be had. Shall we have wise legislation based on good habits of functional thinking? Or is the legislation to be swayed by mere guesswork and emotion—perhaps by prejudice? We must have elections. Some of these

elections will turn on questions of control of public utilities, on bond issues, on pensions, on the tariff. Are the voters to have real opinions? Or are they to be swayed by passion and prejudice, or by the influence of interested parties?

We have heard a great deal in educational discussions of this question of education for citizenship. Many a subject in our curricula bases its whole claim upon the value of the knowledge of that subject as a preparation for proper views on public questions. Thus history is not so much of selfish benefit to the individual; it is of paramount importance to the State. For voters ignorant of the historical development of our institutions, and of the trials and failures of other institutions, would be in grave danger of making fundamental mistakes. Sociology, economics, public law, and other dignified and important subjects derive at least a large share of their value from their connection with public affairs, and with the necessities of training for good citizenship.

This I will not deny, but rather affirm. Yet I may surely add that the most vital questions before the public today are those of a quantitative nature, involving relationships between quantities of no mean order. To wisely vote upon these questions, even to hold a respectable opinion regarding them, must presuppose at least some degree of a habit of functional thinking. Indeed, in our present day problems regarding bond issues, regarding pensions, regarding life insurance, regarding income taxation, regarding control of public utilities, regarding all valuations, history tells us little because our problems are either newly formed or they exist in such unprecedented size that previous experience is not a guide. Even more than history or sociology, mathematics is the key, for these problems are mathematical and functional in character. We can expect little from the great mass of the people on such problems. The danger is that this lack of comprehension of quantitative relationships may lead to disastrous decisions under some wild leadership. Is this unthinkable? Have we not seen as wild leadership and as distorted decision? We need at least a small degree of ability to think about quantities on the part of all citizens. We need leaders, politicians, statesmen, who can think very clearly on such questions, so that they may guide public opinion aright. To train such leaders is a function of the college. I feel deeply

that it is of vital moment to democracy to increase as much as possible the widespread appreciation of quantitative relationships, and habits of thought concerning them. That very remarkable man, H. G. Wells, in his *Mankind in the Making,* has said: "The new mathematics is a sort of supplement to language, affording a means of thought about form and quantity, and a means of expression, more exact, compact, and ready, than ordinary language. The great body of physical science, a great deal of the essential facts of financial science, and endless social and political problems, are only accessible and only thinkable to those who have had a sound training in mathematical analysis. The time may not be very remote when it will be understood that for complete initiation as an efficient citizen of the new great complex world-wide states that are now developing, it is necessary to be able to compute, to think in averages and in maxima and minima, as it is now understood to be necessary to be able to read and to write."

This whole paper might well be said to be an introduction to and a preparation for this very strong statement, which is indeed so strong as to be well-nigh incomprehensible without some preparation. With it I would rest my case. But let me warn those of you who are teachers of mathematics that Wells does not mean our traditional formalized courses when he speaks of the "new" mathematics. He means a mathematics reborn, revitalized, conscious of its soul and of its destiny. His praise is not for smug content with what is commonly thought of when mathematics is mentioned. If we base the case of mathematics on such statements as his, or on those of this paper, or on those of the National Committee which I quoted in the beginning, we must remake our courses, we must abandon many of our traditions, we must shape our work so that it will indeep help to give our students real and correct habits of functional thinking in all those quantitative situations which they will meet in their private lives and in their public activities.

CALCULUS FOR SCHOOLS

By W. H. TYLER

The title of this paper may have various psychological reactions on the reader, according to his mental attitude and experience. To some it will perhaps seem as reasonable as teaching Greek in the kindergarten. A generation ago, when some of us were young, calculus was an ultimate goal of college mathematics, to be attained only after prolonged and painful servitude in the regions of college algebra and conic sections. The idea of teaching it even to college freshmen would have seemed heterodox, if indeed anybody had been so radical as to propose it. When finally reached it appealed only to the algebraically minded with its marvelous intricacies of partial fractions and reduction formulas. Excursions into geometry relieved the perplexities of some, to be sure, but the fact that calculus should be anything but an end in itself was more or less effectually disguised.

The completion of the subject was something to be celebrated with bon fires and illuminations, followed by very prompt oblivion of whatever had been inadvertently assimilated, except of course by the elect few who on the basis of congenital predilection for mathematics—and perhaps unfitness for anything else—were to continue into the fastnesses of differential equations and the like. Why then, in the name of all that is conservative and orthodox in education, talk about calculus for secondary schools?

A partial answer to this may be found in the history of elementary mathematics and its teaching. Our traditional school programs include—and have included for several generations—algebra through quadratic equations and the six books of plane geometry, but not many centuries ago the situation was notably different. In the sixteenth or seventeenth century, for example, education for boys of fourteen to eighteen would have included something resembling our grammar school arithmetic with protracted study of long division and a great deal of time on weights and measures, perhaps the algebra of the first degree equation and a little of the elements of plane geometry, the bulk of which was reserved for university students, together, of course, with trigonometry and the less elementary parts of algebra.

The evolution of the school program, in spite of inertia and powerful conservative tendencies, has consisted in the continual sifting of the inherited program, the progressive elimination of topics of minor importance and the bringing back of the essential parts of new material from the higher grades to the lower. It is unthinkable that this process should have reached the limit of its possibilities in our time. In our algebra, for example, there are many survivals of material which antedates the invention of the calculus and while not without considerable intrinsic interest for the algebraically minded, is of merely historical importance. The solution of the cubic equation, for example, was a great achievement of the sixteenth century and has a just claim on the interest of present day teachers of algebra, but that alone would be no sufficient reason for teaching it even in college algebra.

Again, in geometry, the Euclidean development while not inappropriate for an age when it constituted the whole body of mathematical training and when physics, chemistry and biology were unknown or unorganized, contains much that is non-essential in the modern development of mathematics.

Under favorable conditions at the present time there can be no fundamental difficulty in thoroughly teaching the essentials of arithmetic, algebra and geometry to boys and girls by the time they are sixteen. The question then is, as to the best use of available time for a year or two more, first as between mathematics and other subjects; second, as between various branches of mathematics. Is it best for a boy or girl to devote another year or two to mathematics, to spend it on additional algebra or on elementary analytic geometry, or on elementary calculus, or on some branch of applied mathematics, as simple surveying, etc.? No one answer to this question will be uniformly valid but in many cases the answer should unhesitatingly be the choice of the elementary calculus, particularly for pupils who are *not* continuing their education through colleges or scientific schools where the calculus would be inevitable.

The main reasons for this contention are:

First, the immense importance of the fundamental notions of limit and derivative without which a person is unable to form a clear and distinct conception of quantitative change in the world around him except in the crude sense of averages;

Second, the simplicity of the technique, if attention is confined to the polynomial functions with perhaps a few additions from the fractional or trigonometric;

Third, the importance of the notion of integration as a summation, and

Fourth, the comprehensiveness of calculus as built upon all the preceding mathematical subjects, and, on the other hand, as opening up unlimited possibilities of concrete application.

Granted all that has been said as to the value and importance of elementary calculus, is it practicable to teach it under secondary school conditions? This question must, of course, be finally answered on the basis of actual experience (to which the present writer can make no claim). Fortunately, however, the data on this side are fairly abundant. Elementary calculus has long been taught in the French lycée and the German gymnasium. It is now taught in English schools to boys of corresponding age. Of course it is necessary to adapt the teaching and the text to the age and capacity of the pupils.

These generalities may seem an inconclusive preamble without something more specific—which should, of course, presuppose experience. As to this, it is submitted that while there is doubtless a marked difference in capacity between the average high school senior and the average college freshman, there is no such gulf fixed between the best tenth—let us say—of the school seniors and the average freshman. It is for that tenth in the larger and stronger schools that the elements of the calculus may well be offered.

This does not mean a series of technical terms, elaborate definitions and rigorous proofs. Statements must be carefully framed and solecisms severely excluded, but the point of view may be largely intuitive and formal proofs subordinated or postponed. It is essentially a matter of studying the change in one quantity due to the change in another—and making that study more searching and exact than it can be made without the calculus.

A familiar—possibly too familiar—illustration is the automobile trap. The policeman knows no calculus, but he is well aware that the speed of a car at a station A in a distance S is not to be determined by merely dividing the total distance by the time T in which it was traversed. He must take a short interval

$AB(=\triangle S)$ and divide this by the corresponding short time $(\triangle t)$ in which *it* was covered. Even this will be only an average, less than the speed at A if, for example, the driver chanced to observe the policeman in time. The true speed at A can only be defined as the limit $\triangle s / \triangle t$ as t o. The policeman is seeking a derivative—however innocently.

Again, in physics a boy learns that the motion of a stone thrown at an angle of elevation is expressed approximately by the equations $y = v_0 t \sin a - \frac{1}{2}gt^2$, $x = v_0 t \cos a$. How high will it rise, at what distance will it strike the ground and for what value of a will that distance be greatest. The possibility of solving the problem more laboriously without calculus detracts nothing from the value of the illustration.

The other great calculus concept of integration as a summation can be based on the study of simple problems in pressure upon vertical as contrasted with horizontal surfaces. If P is the total pressure above a certain horizontal line at a depth y, $\triangle P$ the pressure on the adjacent horizontal strip of width $\triangle y$ and area $\triangle A$ then $\triangle P$ lies between $Ky \triangle A$ and $K(y + \triangle y) \triangle A$, K denoting the density, so $P = K \int y\, d A$. The summation by integration will naturally have been approached by preliminary approximate summation without calculus, for example, finding the area of a river cross-section by the trapezoids which correspond to soundings at definite intervals. Much emphasis will be put on simple concrete problems in maxima and minima, selected with a view to avoiding algebraic complications. Contrast the value and interest of such work with the ordinary topics of advanced algebra, for which even the mathematician will rarely have future use. The study of the elementary calculus clarifies the idea of dependent variability—functionability—as nothing else can, and that idea pervades the quantitative activities of life.

THE PSYCHOLOGY OF PROBLEM SOLVING*

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In a previous article we called attention to some of the differences of opinion concerning the application of algebraic technique to the solution of problems. In the present article we shall take up systematically the questions there suggested, together with others, and present the results of certain investigations which we have made to aid teachers in deciding what to do with problem solving, when to do it, and how to do it.

At the outset we need to distinguish certain types of work all of which might with some justification be called applications of algebraic technique to the solution of problems, but which differ notably in the psychological demands they make of a pupil, in the psychological effects they have upon him, and in their uses in the algebra course. All are different from mere computation, evaluation, reading or making graphs, or the solution of equations already framed.

The more important types are shown below.

Type I. Problems to answer which no explicit equation or formula is needed or supposed to be used.

1. Concerned with knowledge of meanings, *e. g.*, of literal numbers, negative numbers, exponents.

   If pencils cost c cents each what will n pencils cost?

   Express 248 ft. below sea level if sea level is called 0.

   If $a \times a \times a \times a$ is a^4, how will you express the product of a row of n a's?

2. Concerned with knowledge of operations.

   What was your average score in a game in which you made these separate scores: -8, $+4$, -6, -2, $+7$?

3. Concerned with combinations of meanings and operations or with other aspects of algebra.

   Under what conditions will $a = \frac{1}{a}$?

   In $a = p + \frac{q}{r}$ what will be the effect upon a of an increase in r?

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*The investigations on which this article is based were made possible by a grant from the Commonwealth Fund.

The value of v_2 for $v_1 = 4$ was omitted from this table by the printer. What do you think it was, approximately?

If	v_1	is	0	v_2	is	4
"	"	"	1	"	"	5.02
"	"	"	2	"	"	7.96
"	"	"	3	"	"	12.98
"	"	"	4	"	"	.....
"	"	"	5	"	"	29.03
"	"	"	6	"	"	40.05

Type II. Problems to answer which an equation or formula is supposed to be used.

A. The equation or formula is one of a group of known formulae.

1. Which of these formulae fits the problem is also known.

$F = 32° + \frac{9}{5} C.$

What does 86° F equal on the Centigrade scale?

2. Which of them fits the problem is not known. Pupil must select the formula as well as fit the facts properly to it.

The meanings of these are known. $v = at$, $v = gt$, $S = \frac{1}{2}at^2$, $S = \frac{1}{2}gt^2$, $v = \sqrt{2as}$, $v = \sqrt{2gs}$, $g = 32$.

Neglecting the resistance of the air, how far will a bullet dropped from a height of 5000 ft. fall during the sixth second?

B. The equation or formula is not known, but must be constructed by the pupil.

a. The equation or formula is primarily a rule for all cases of a certain relation.

Frame equations for finding the dimensions of a rectangle twice as long as it is wide to be of *a* sq. in. area. Find the dimensions when *a* is 10. When *a* is 20. When *a* is 100.

b. The equation (not usually called a formula) is primarily an organization of data to secure the result in one special case.

A girl wishes to have a rectangular card twice as long as it is wide and 10 sq. in. in area. How long shall she make it?

The psychology of the I-1 type is clear. Such problems, carefully chosen and graded, may be used very helpfully to teach meanings and to test, strengthen, extend and refine knowledge of meanings. Teachers of algebra should study the problem material of this sort devised by Nunn [1913] and by Rugg and Clark [1918].

Problems of the I-2 type, which apply algebraic computations in useful ways, are very rarely used. This may be because teachers think there is no need of applying computation; or it

may be because genuine uses for algebraic computations are not to be found in such matters as a first-year high-school pupil can understand.

The first reason is almost certainly a bad one. Pupils, save some of the very intellectual, are stimulated by seeing what a computative procedure is for, and by associating it with the world outside of mathematics. In arithmetic it is found serviceable to introduce each new item of computational method by some genuine and interesting problem whose solution is facilitated by the computation in question. Students of algebra can doubtless get along without such stimulants better than students of arithmetic, who are younger and duller; and may gain less from them. But they, too, will gain much from such introductory problems showing the service which the computation performs. Again the teacher should examine the problem material of this sort in Nunn and in Rugg and Clark.

The second reason is in part valid. Much of the computational work often done in courses in algebra cannot well be introduced by or related to problems from the world the pupil knows. The problems that one might invent would not make the computation clearer or easier or more esteemed or longer remembered. The value of such computation is questionable.

The miscellaneous group listed as I-3 represents a borderland between what we ordinarily call problems and tasks calling for mathematical inference and conclusions of all sorts. Too little attention has been paid to this group by teachers of algebra. There has been, in fact, so little of such work that we can hardly judge of its value; but it would at least add variety, give more scope for "original" thinking, and assist in integrating a pupil's algebraic abilities into something which for lack of a better term we may call an "algebraic sense"—a somewhat general readiness to see algebraic facts and to think about them with all his algebraic equipment.

The work of II-A represents what is regarded by teachers of the physical and social sciences as the most essential contribution of algebra in preparation for their study. It has the merit that the facts operated with have some chance of being themselves worth thinking about—are not mere valueless items about A's age, or B's time in rowing up a stream, or C's buying and

selling of sheep; and that the relations amongst these facts are likely to be important relations in nature; and that the results obtained are such as a sane man might obtain in that way.

It has also the merits that it emphasizes the fact that a letter can be used to mean any one of the class of numbers that fulfil certain conditions, and that it leads up to the general treatment of the relation of one variable to another.

Two cautions are useful in connection with such work. The first is to be careful not to burden pupils unduly with learning physics or astronomy or engineering for the sake of having genuine formulae. Formulae whose meanings are obvious from a careful reading should be preferred. They should not be complicated by unfamiliar terms, or by the need of difficult inferences to secure consistency in units. Tabular and graphic work may be usefully coordinated with the problem work. The same formulae may often be used in exercises in formula reading and formula framing, evaluation, and the understanding of relations, before they are used in connection with verbal problems.

The problem material from science and engineering that can be adapted so as to satisfy wise teachers of algebra that it is as good for their purposes as the material in customary use about familiar facts like the ages of boys, the hands of clocks, boats on streams, or tanks and pipes, may be rather scanty. The question then arises of using made-up formulae, such, for example, as:

A boy's father gives him each month half as much as the boy earns.

1. Let e = what the boy earns in any month. What will equal what his father gives him for that month?

2. Let T = what the boy gets in all that month. Make a formula for finding T.

3. What will the boy receive in all in each of these months: January, when he earns $10? February, when he earns $12? March, when he earns $8.60?

4. How much must he earn in a month to get (in all) $20 that month?

It is for such cases that we need the second caution, namely, that we avoid in such made-up formulae the unrealities and trivialities that have characterized so much of the problem material of the past.

We have left the II-B type where the equation or formula is not known but must be constructed by the pupil. This is problem material par excellence, and is, in fact, all that certain teachers would consider worthy of the name. Within it, the II-B b type is in far wider use than the II-B a type. In the rest of this chapter, consequently, unless the contrary is specified, we shall mean by a problem one of the type of II-B b where an equation is built up, organizing the data given in the problem about some one state of affairs, so as to secure the answer to one or more quantitative questions about that particular state of affairs.

We shall deal with the following matters, in the order given:

The genuineness of the problem.

The importance of the problem.

Shall every technique be applied to problems?

How far shall the problems be worked as originals, and how far shall routine procedures for solving a certain kind of problem be taught?

The overestimation of the educative value of the verbal problem.

The use of problems at the beginning of a topic to show the need for certain technique and to facilitate the mastery of the technique, as well as at the end to test the ability to apply the technique.

Criteria in selecting problems.

Problems as tests.

Real versus described situations.

Isolated and grouped problems.

Problems requiring the selection of data, as well as their organization.

Problems requiring the discovery of data as well as their selection and organization.

Problems requiring general solutions.

Problems of puzzle and mystery.

The election of problems by students.

Genuineness

Relatively few of the problems now in use are genuine. First of all, over half of them are problems where in the ordinary course of events the data given to secure the answer would them-

selves be secured from the knowledge of the answer. For example, "In ten years John will be half as old as his father. In twenty years he will be three-fifths as old as his father. How old is John now? How old is his father?" In reality such a problem would only occur in the remote contingency that someone knowing that John was 10 and his father 30, figured out these future age ratios, then forgot the original 10 and 30, but remembered what the future ratios were!

We have made the count for three representative textbooks of excellent repute with the results shown in Table 1.

TABLE I

Percentages of "Answer Known" Problems.*

	Up to the beginning of fractions.	The beginning of fractions up to quadratics.	Quadratics and beyond.	Total
Book A	52	69	54	57
Book B	45	36		42
Book C	53	52	51	52

Such problems, if defensible at all, are defensible as mental gymnastics, and as appeals to the interest in mystery and puzzles. As such, they are better if freed from the pretense at reality. "I am thinking of a number. Half of it plus one-third of it exceeds one-fourth of it by seven. What is the number?" is better than problems which falsely pretend to represent sane responses to real issues that life might offer. It is degrading to algebra to put it to work searching for answers which in reality would have been present as the means of framing the problem itself, save frankly as a mere exercise in sharpening one's wits and in translating a paragraph into an equation.

Of the problems which are not clearly ruled out by this criterion many concern situations or questions or both which are not genuine, because in the real world the situation would probable not occur in the way described or because the answer would not be obtained in the way required.

We can make a scale for genuineness running from problems that are fantastic to problems that are entirely genuine. We

*An "Answer Known" problem is one where it is highly probable that in real life the data given would be obtained from the answer rather than the answer from the data. The totals from which the percentages are computed include problems of Type I and Type II-A. If only II-B problems were considered, the percentages of "Answer Known" problems would be much higher.

may set as a very charitable criterion that a problem should be as high as 4 on this scale, 4 being the average genuineness of the following problems.

REALITY 4

A. Three men are asked to contribute to a fund. The first agrees to give twice as much as the second, and the third agrees to give twice as much as the first. How much must each contribute to make a total of $1050?

B. What angle is five times its complement?

C. A boy knows that his boat can go 6 miles per hour with the current and 3 miles per hour against the current. How far can he go and return making the whole trip in just 3 hours?

D. The diagonal of a rectangle is 102 inches and the base of the rectangle is three times its altitude. What is the length of its base?

E. The principal varies directly as the interest and inversely as the rate. If $2000 brings in $125 interest at 4%, how much principal will yield $500 at 5% for the same time?

Reality 4 is obviously not a high standard. It is doubtful whether in all the world's cases of conditional giving the problem of A has ever occured. If B has ever occurred it probably has been in such connections that the immediate solution by $180° - \frac{1}{6}$ of $180°$ would be used. C illustrates a genuine relation but one which in reality is usually so complicated by other circumstances that only approximate estimate is made; hence the equational treatment seems rather pedantic. It is very hard to conceive cases where a person would know the proportions of a rectangle and the length of its diagonal and not already know the length of the base. The determination of the investment required to yield $500 when you don't know for how long, but do know that $2000 at 4% brings in $125 in the time in question, would not occur probably once in the lifetime of a million men.

We have counted for one of the books noted above the number of problems which passed the "answer known" criterion but failed to rate as high as 4 for genuineness. There were 70 out of 213.

Importance

Within the minority of problems that remain after the exclusion of bogus problems and problems whose "genuineness" is less than 4, a considerable percentage concern matters that are of importance to few people and not of much importance to them. The problems of the book in question were rated by four psychologists, three of whom were well versed in mathmatics. Importance 3 means the degree of importance possessed by the following:

IMPORTANCE 3

A. Divide $108 between A and B so that A receives eight times as much as B.

B. A has $2000 invested at 4%. How much must he invest at 6% to make the total yield equal to 5% on the total investment?

C. Mr. A paid $300 per share for some stock. At the end of five years he sold it for $550 per share. What rate of simple interest did his money produce for him during the five years?

D. Mr. A can plow a field in 6 days. Mr. B can plow it in 9 days. How long will it take them to plow the field if they work together?

E. A man does one-third of a piece of work in 10 days. He and another man finish the task together in 8 days. How many days would it take the second man to do the work alone?

F. ABC is an isosceles triangle. AD is its altitude, AD being perpendicular to BC. BD = DC. If AB is 18 inches and BC is 15 inches, find AD.

G. If a boy earns $520 during his first year of work and receives an increase of $50 a year each year thereafter, what salary does he receive the tenth year? How much has he earned in all during the ten years?

H. Each year Mr. A saves half as much again as he saved the year before. If he saved $64 the first year, how much will he save in all in seven years?

If we eliminate all the problems whose importance is less than this, we have left 96, about one-fifth of the total list. Of those so left, many are more suitably solved by mere arithmetical computation without any equation than by the organization around a symbol for the desired number and an equality sign. Omitting these also we have left 61 of the original 491.

These genuine problems which pass our minimum standard of importance for life seem worthy of presentation. The problems themselves show the teacher's available resources in this respect better than a description or tabulation of them would.

Some of these 61 are clearly problems of Type I where no equation or formula is needed (*e. g.*, One bu. equals 32 qt. How many qt. in 4 bu.? In x bu.?) Others are clearly of Type II-A-1 where only substitution in a formula presented at the time is needed (*e. g.*, changing Fahrenheit temperatures to Centigrade). Omitting these, we have left 49, one-tenth of the original series. These 49 problems appear on pages 221 to 223.*

Table II shows the facts separately for the work up to the beginning of fractions, from there up to quadratics, and from quadratics on.

*The problems given here are not quotations, since it seems desirable to preserve the anonymity of the source, but are duplicates of the originals in general nature and form.

TABLE II

Analysis of the Verbal Problems of a Standard Textbook.

	Up to the beginning of fractions.	The beginning of fractions up to quadratics.	Quadratics and beyond.	Total.	
1. Total number of verbal problems	268	126	97	491	
2. Numbers in which the answer would ordinarily not have to be known in order to obtain the data of the problem	129	39	45	213	(43%)
3. Number of these (2) which are rated as 4 or above for reality	100	16	27	143	(29%)
4. Number of these (3) which are rated as 3 or above for importance	73	9	14	96	(20%)
5. Number of these (4) which are not much more readily solvable by arithmetic alone and which are not clearly of Type I (where no explicit equation or formula is needed)	39	9	13	61	(12%)
6. Number of these (5) which are not clearly of Type II-A, where only substitution of numbers in a given formula is required	29	9	11	49	(10%)

An analysis like that of Table II has been made for a second book of excellent repute, which contains work only up to quadratics. It gives the following results:

	Up to fractions.	Fractions to quadratics.	Total.	
1.	283	129	412	
2.	157	82	239	(58%)
3.	125	56	181	(44%)
4.	121	23	144	(35%)
5.	39	19	59	(14%)
6.	38	15	53	(13%)

Eighteen of the problems in this second book left in class 6 are not typical problems of Class II-B. Ten are constructions of graphs, and eight are applications of simple trigonometrical facts not usually taught in algebra hitherto.

The Highest Ranking Tenth of Problems in Respect to Genuineness and Importance

1. The selling price of this book is five-fourths of its cost. Find its cost if it sells for \$2.00.

2. In making a certain casting $1\frac{1}{2}\%$ of the metal is lost in the melting. How much metal is needed to make a casting weighing 86 pounds?

3. Cotton seed meal is used as a fertilizer. It contains approximately 7% of nitrogen. If a farmer wishes to put 15 pounds of nitrogen on a certain field, how much cotton seed meal must be purchased?

4. Tobacco stems contain about 8% of potash. How many pounds of tobacco stems must be bought to obtain 12 pounds of potash?

5. Divide \$108 between A and B so that A receives 8 times as much as B.

6. Three men are asked to contribute to a fund. The first agrees to give twice as much as the second and the third agrees to give twice as much as the first. How much must each contribute to make a total of \$1050?

7. The minimum temperature on February 2nd at Minneapolis was — 15; the maximum was — 4. What was the range of temperature there of that day?

8. Mr. A wishes to enclose a rectangular field, 20 rods wide. He wishes to make the field as long as he can, using 214 rods of fencing. How long can he make it

9. If the cost of a car is p dollars and the rate of gain is 20%, what is the gain? What is the selling price? $(p + .20p = ?)$

10. What was the cost of a car sold for \$13.20 if the gain is 10%?

11. Mr. A wishes to make 20% on some chairs. At what price must he buy them if he is to sell them at \$2.00 each?

12. Mr. B wishes to sell chairs at \$7.00 each. At what price must he buy them so as to make 12% on the cost?

13. Mr. C knows that he can sell a piece of property for \$3540. How much must he pay in order to make a profit of 18%?

14. What principal must be invested at 4% to yield an income of \$600 a year?

15. How long must \$2000 be invested at 5% simple interest to produce \$375 interest?

16. The amount equals the sum of the principal and the interest. Express the amount at the end of a year when p dollars are invested at 4%.

17. What is the amount at the end of two years if p dollars are invested at 5%?

18. What sum of money invested at 6% simple interest for three years will amount to \$4000?

19. How long will it take \$1000 to amount to \$1500 if it is invested at 5%? $(1500 = 1000 + 1000 \times .05y.$ Solve for y.)

20. How long will it take \$1000 to double itself at 6%?

21. Let A represent the number of dollars in the amount. Let P, R and T have their usual meanings. Show that $A = P + \frac{PRT}{100}$

22. Use the formula above to find how many years will be required for \$7000 to amount to \$9100 at 5% simple interest.

23. Use the formula to solve this problem. Mr. A paid \$300 per share for some stock. At the end of 5 years he sold it for \$550 per share. What rate of simple interest did his money produce for him during the five years?

24. Mr. A has tea worth 65 cents and tea worth 45 cents per pound. How many pounds of each should he use to make a mixture of 100 pounds to sell at 53 cents a pound?

25. Mr. B has tea selling at 70 cents a pound and tea selling at 50 cents a pound. How many pounds of each should he use to make a mixture of 50 pounds selling at 62 cents a pound?

26. Same as the two previous, using different numbers.

27. Mr. A has $2000 invested at 4%. How much must he invest at 6% to make the total yield equal to 5% on the total investment?

28. Mr. B has $6000 invested at 3½% and $9000 at 4%. How much must he invest at 6% to make the total yield equal to 5% on the total investment?

29. A boy knows that his boat can go 6 miles per hour with the current and 3 miles per hour against the current. How far can he go and return, making the whole trip in just 3 hours?

30. A man can do a piece of work in 8 days. What part of it can he do in one day? In 7 days? In x days?

31. A man can do a piece of work in x days. What part of it can he do in 1 day? In 5 days?

32. A can do a piece of work in 6 days. B can do it in 10 days. How much can A do in one day? In x days? How much can B do in one day? In x days? How much can A and B together do in one day? In x days? How much can A do in 2 days? How much can B do in 5 days? How much can A and B together do if A works 2 days and B works 5 days?

33. A can do a piece of work in 10 days. B can do it in 5 days. How long will it take A and B together to do it?

34. A can do a piece of work in 8 hours. B can do it in 24 hours. How long will it take A and B together to do it?

35. Mr. A can plow a field in 6 days. Mr. B can plow it in 9 days. How long will it take them to plow it if they work together?

36. One machine can do a piece of work in 4 hours. Another machine can do it in 6 hours. How long will it take them both together to do it?

37. A can do a piece of work in 15 hours. B can do it in 18 hours. If A works for 7 hours, how long will it take B to complete the work?

38. A does one-third of a piece of work in 5 days. A and B complete the job by working together for 4 days. How long would it take B to do the job alone?

39. ABC is an isosceles triangle. AD is its altitude, AD being perpendicular to BC. BD = DC. If AB is 18 inches and BC is 15 inches, find AD.

40. If a boy earns $520 during his first year of work and receives an increase of $50 a year each year thereafter, what salary does he receive the tenth year? How much has he earned in all during the ten years?

41. On January 1st of each of 10 years a man invests $100 at 5% simple interest, what will principal plus interest amount to at the end of the tenth year?

42. Mr. A owes $2000 and pays 6% interest. At the end of each year he pays $200 and the interest on the debt which has accrued during the year. How much interest will he have paid off when he has paid off the debt?

43. Mr. A is paying for a $400 lot at the rate of $20 a month with interest at 6%. Each month he pays the total interest which has accrued on that month's payment. How much money, including principal and interest, will he have paid when he has freed himself from debt?

44. Mr. A plans to give his son 10 cents on his fifth birthday, 20 cents on his sixth, and each year thereafter to the eighteenth birthday, inclusive, to double the gift of the preceding year. How much will this be in all?

45. A problem in finding the height of a tower by similar triangles. A diagram is given.

46. Finding the width of a pond by similar triangles, and subtraction. A diagram is given.

47. Finding the width of a pond by similar triangles and double subtraction. A diagram is given.

48. The number of tiles needed to cover a given surface varies inversely as the length and width of the tile. If it takes 300 tiles 3 inches by 5 inches to cover a certain surface, how many tiles 4 by 6 will be needed for the same area?

49. The number of posts needed for a fence varies inversely as the distance between them. If it takes 120 posts when they are placed 10 feet apart, how many will it take when they are placed 12 feet apart?

It must be confessed that this list of what one of our standard instruments for teaching algebra offers as genuine problems to be solved by framing an equation does not support the general high estimation of problem solving of the II-B-b type. Problems 16 to 23 and 41 will not be acceptable to many because they neglect the fact that in real life the interest on the investment is almost always paid at stated intervals, not when the principal is repaid, and so can be compounded by reinvesment. Nos. 6, 27, 28 and 43 are rather fantastic. Nos. 48 and 49 require a method of finding the number of articles required which would rarely be wise, and never necessary, to use in such situations. Of the other problems, some are very probably better dealt with by the arithmetical methods which the pupils have already learned to use in such cases.

The advocate of the made-up problems will use the scantiness of this list as an argument that we must resort to the made-up, even insane, problems in order to give sufficient practice in applying principles and technique. But why should we give any practice in applying a principle or a technique to created problems when there are no sane problems to which it applies? Moreover, we must not assume that all the problem material which is genuine and of a fair degree of importance has been collected. At first thought, it would seem probable that it had, since for at least a decade progressive teachers and textbook makers have been fully aware of the need for it. A closer study of the matter, however, reveals that ingenuity and inventiveness and careful investigations do bring returns here as elsewhere. Many more such problems appear in the textbooks and teaching of

1920 than were available in 1900. Nunn has made very notable contributions. We may hope that the Nunns of the future will add more. Until we have canvassed the world's work thoroughly for problems that are genuine and important, we ought not to turn to those that are artificial and trivial.

It is a modern tendency to extend the list of genuine problems by teaching certain facts of physics, engineering, astronomy, navigation and the like so as to secure material for practice with the applications of algebra.

We very much need measurements of the time-cost of this, and of its effect upon interest in the sciences in question, in typical cases. The expectation is that often the game is not worth the candle unless it is very skilfully played, and that an undesirable attitude toward science may often result. It should be noted that the experts in teaching science rather carefully avoid algebraic and other quantitative work for pupils in high schools. High-school teachers of chemistry, geology, physical geography, the biological sciences and economics are cautious about employing anything mathematical beyond the simplest; and this partly because they fear that it will repel students. Even in physics, descriptive work is emphasized rather than the fundamental equations; words are used instead of symbols, and sentences instead of formulae. This in spite of the fact that physics is taught in the last or next to last year of high school to a select and mature group. There is a danger that when we select problem material for algebra from the sciences we may be burdening algebra with the least attractive features of science and penalizing science by displaying its least attractive features to the pupil at the beginning of his high-school course.

There has been, so far as I am aware, no direct observational or experimental evidence published concerning the reactions of pupils to these problems taken from the sciences. Nor have we found facilities for securing such. We have, however, secured the judgments of the four psychologists mentioned previously.

They rated seventeen such problems (11 about the principal that weight times length of lever arm equals weight times length of lever arm to make a balance, 5 about freely falling bodies, and 1 about the pressure-volume relation in gases) for reality, importance, interest, value in showing and in applying mathematical laws, excellence of statement, and value in teaching facts

or laws outside of mathematics. In the combined weighted average, these problems from physics were somewhat above the average of problems in present-day textbooks.

Selection of Techniques for Application

It is hard to find psychological or pedagogical justification for the custom of concluding each topic in algebra by a series of verbal problems whose solution requires the operation of the mathematics taught under that head. The custom seems to be due partly to habits carried over from arithmetic, partly to the general fondness of intellectual persons for neat symmetrical systems, partly to a general overvaluation of the verbal problem as a means of mental training, and partly apparently to an insufficient appreciation of pure mathematics itself.

It is not likely that the arrangement of problems applying mathematical technique which is best for all children in grades 3 to 6 will be the best for the third of them who go on to study algebra in grade 9. Nor is it at all certain that "technique—application—technique—application" is the best arrangement in grades 3 to 6. Good practice in the teaching of arithmetic now supplements this arrangement in grades 3 to 6 by an arrangement by topics like "Earning and Saving," "Distances in a City," "House Plans," "A School Garden." In grades 7 and 8, the arrangement has long been largely by topics like Insurance, Investments, Interest Given by Savings Banks, and Bank Loans, and the like, and is developing toward an arrangement by topics like Food Values, City Expenditures, and Wage Scales.

Whatever arguments may be derived from the advantages of system would seem to be in favor of giving the main treatment of problem solving in one large unit, the general task of which would be to show that any number or numbers which can be found from certain given data, can be found by expressing the proper data in an equation or equations and solving. This chapter could well come after the pupil had learned to add, subtract, multiply and divide with literal numbers, including such simple fractions as should be mastered, and before the systematic treatment of the relation $y = ax + b$, or any treatment of quadratic equations. If a problem is suggested that leads to a quadratic (or a cubic), no harm will be done. The pupil may frame the equation, and leave it for solution until he learns the tech-

nique. This matter will be discussed further as one special problem of the order of topics in algebra. Our present purpose is simply to suggest that system does not require, or even favor, applying every technique indiscriminately in verbal problems. Of the general over-valuation of verbal problems and undervaluation of pure mathematics we shall treat in detail later.

All these are matters of minor importance for our present question if we accept as true a proposition which seems to the psychologist almost indubitable; namely, that the peculiar educative values of these verbal problems are attained by *framing* the right equations, solving them being not very greatly different from solving a similar equation framed for you by the textbook. If the problems are given primarily to train the pupil to frame the right equation or equations, we care very little about what computational techniques they happen to lead to. To take the extreme case, suppose that pupils only framed and *never* solved the equations, as in the Hotz test for problem solving. It would then be of almost no importance which techniques were required in these solutions—whether, for example, abilities with surds, quadratic equations, and certain factorizations were or were not applied. In so far as the peculiar value of problems is in framing the equations it is better *not* to give, after each technique is learned, many of the problems applying to it, because this tempts the pupil to expect that the problems will have a certain sort of equational form. He is tempted to work *toward* a certain sort of equation, instead of *from* the data given. It would then be amongst "miscellaneous" that a problem usually gave its best training.

We do not mean to imply that the framing of an equation and its solution are as educative if done a month apart as if done together, or that solving equations already framed for you is as educative as solving equations which you have framed yourself. We do claim that the peculiar virtue of the verbal problem is in the framing, not the solving, and that problems should be selected and arranged from this point of view rather than as exercises to show that certain algebraic computational tasks can be used in problems and to give practice in their use.

Problems as Originals and as Semi-Routines

The guiding principles in relation to this question can be briefly stated as follows:

Other things being equal, it is more educative to solve a problem as an original. Individual differences in ability need most of all to be allowed for when problems are given as originals. It is not probable that a pupil's efforts to solve problems are of great value to him when he fails with more than two out of three of them. On the other hand, pupils who are able to solve a certain type of problem as an original should certainly be excused from training in a routine method of solving it. Special training in the method of solving a certain type of problem is not desirable unless the problems in question are genuine and of some considerable importance. Clock, digit, age and other similar problems should be given as originals, if at all. Whatever view we take of the amount of general ability developed by problem solving, one of the best ways to develop it is by trying to solve problems as originals, and, in case of failure after a reasonable effort, being given such assistance as enables one to solve them.

(To be concluded in the May issue.)

CULTURAL VALUE OF MATHEMATICS

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"The intellect never slumbers," but is ever searching for knowledge and truth. It is ever groping about in the darkness of error and of doubt, and if truly honest in its search, uses every light which an all wise and loving Master has given it to detect the slightest flaw in every finite problem which is presented to it. Its restlessness continues until the goal of absolute certitude is reached.

Perhaps an alluring invitation urges that intellect to turn aside into a by-path easier and more attractive than the rough hard road of truth, which appeals only to him who is morally and intellectually rugged. If so, its faithful friend philosophy calls out to it reminding it that fact is the beginning and end of all things, that fact will not, and cannot be brushed aside. That whether in a purely intellectual pursuit or in the stern realities of an every-day common-place struggle for existence, "Truth is mighty and must prevail."

It reminds that intellect that to every manifestation of effort or action in a substance there is a proper prôportioned object on account of which this power is put forth in its specific activity. That effect is always consequent upon a cause, and that that intellect seeking enlightenment must first of all, recognize itself as a created thing, and an effect of a great cause which lives beyond the finite, and without which nothing could be made that was made.

The human intellect made therefore for truth is tireless in its search for knowledge. This is its instinctive quality and is as constant and unvarying as the daily rotation of the earth in its annual journey around the sun. "The heart was made for God," says St. Augustine, "and it cannot rest until it rests in God." As God is truth, this restless struggle of the intellect against error is but the carrying out of the divine law which reminds him who seeks truth that he shall find his Master.

Traveling in the foot-prints of philosophy is education, another of the hand-maids of the intellect. "Could it be made perfect it would fill the world with truth, goodness and beauty, which are the substance of ideals and ever lure the noblest souls to heroic striving and enduring." (Spalding.)

Mathematics is one branch of education which by its very nature seeks this form of perfection. It treats not of the approximate or problematic, but of the exact measuring of quantities or magnitudes, of the exact ascertainment of their properties and relations, and of unvarying methods by which, in accordance with these relations, quantities sought are deducable from others, either known or supposed. It is the science that deals with absolute truth, having for its function to develop the consequences involved in the definition of a group of a mathematical conception, which conceptions in turn have been definitely and absolutely determined by a fixed number of specifications.

To use a familiar example 2 plus 2 makes 4. No matter where we may be, no matter what the application, or what changes of circumstance, of surroundings, or of conditions may be operating, experience has taught us that no exception exists but that 2 added to 2 always makes 4.

This simple mathematical axiom, typical of the science in its most intricate as well as its least complicated workings, appeals to the tender little child counting his shells and pebbles on the shining sands that line the sea, his untutored mind gradually developing from the darkness of an all but animal ignorance and awakening to the fact of numbers. It appeals with equal strength to the magnate of uncounted wealth, sitting gray and feeble in the chimney-corner of life, the shadows growing longer and longer behind him as his palsied hand still toys with his shells and pebbles, now grown into dollars and cents, into dividends and interest. It is the diapason of the musician, producing harmony, universal concord, correct pitch, in his compositions. It is the square and compass of the master-builder dropping for him his plummet, directing his circles, and calling in his angles in order that a magnificent piece of architecture may rear its head above the earth to glorify God through the handiwork of one of his creatures. It guides the eye and directs the nimble fingers of the designer as the intricate mosaic shapes itself with kaleidoscopic beauty at the command of his genius. It stands by the side of the chemist, watching the beam of his balance, jealously counting each miniim and each grain, as he strives to wrest some secret which jealous nature still guards,

locks within her bosom. It accompanies the physicist to his laboratory, the biologist to his microscope, the astronomer on his journey to the uncharted heaven and ever and always it cries out "Truth is mighty and will prevail," "2 plus 2 makes 4."

In the crypt of St. Paul's in London is the tomb of Sir Christopher Wren, its architect. On the wall above is a tablet with the inscription containing the celebrated words, "*Lector si monumentum requires—circumspice.*" Well indeed may one look at the cultured world and proclaim it a testimonial to that science, which, both in its abstract consideration and its utilitarian applications, has contributed so much to the comfort and happiness of man, by raising his ideals to the advancement of civilization.

We do not speak of the "cultured world" using culture in the narrow sense in which the term is usually applied. That application is the direct antithesis of everything that science or mathematics would seek to accomplish. It is at best provincial, and, with almost no exception, it is artificial and rests on a foundation of untruth and pretence. The culture to which we refer is that development which results from the training and strengthening of all man's powers, mental and physical, with the consequent improvement or refinement of his mind, his morals, his tastes, and all his physical faculties, to the end that he may better fulfill his mission on earth by more freely discharging his duties to his fellow-creatures, to himself, and to his Creator.

Mabie reminds us that "culture is never a taking on from without, of some grace or skill or knowledge, it is always an unfolding from within into some new power, the flowering of some quality hitherto dormant, the absorption of some knowledge hitherto unappropriated. The essence of culture is not possession of information as one possesses an estate, but absorption of knowledge into one's nature so that it becomes bone of our bone and flesh of our flesh. It means the enrichment and expansion of the personality by the taking into ourselves of all that can nourish us from without. Its distinctive characteristic is not extent but quality of knowledge, not range but vitality of knowledge, not scope of activity but depth of life."

A surprisingly large part of all education which men get one from another is moral, not intellectual. Education as such, may consist merely in the acquisition of facts, the storing of them in the strong box of a brain instead of in the musty tomes of a library. Intellectual education demands that this acquired treasure of facts be put to work, that it shall unselfishly toil for others. It is this useful intellectual education which we call culture, not the counterfeit tawdry tinsel of fruitless knowledge. The education which reminds us that we are, after all, our brother's keeper, and that each act, each thought of ours, must somehow, someway work either for weal or woe upon some other creature, or group of creatures, placed by a common Father in our keeping.

Matthew Arnold has made the aim of culture not merely to render an intelligent being more intelligent, to improve our own capacities to the uttermost, but, "to make reason and the Kingdom of God, prevail." He makes culture an impelling force which stimulates the desire, not only to see things as they are, but rather, by the moral endeavor to know and understand more and more the universal order by which all created activity is regulated, and by this knowledge and understanding to conform to it ourselves, make others conform to it, and in this way to help to make the will of God prevail in us and around us.

The aim of true culture therefore, is not alone the securing of the best knowledge, the most accurate science, but to make them tell on human life and character, to the end that human nature in all its capacities may more closely approach perfection.

> "Unless above himself, he can
> Erect himself, how poor a thing is man."

Mathematics is not artificial. It is absolutely true, possessing beauties in its truths, and waging war ever and always, relentless war against untruth. It dwells not in the twilight zone of compromise, but in the noon-day glare of fact. The mind unfolds and develops when it reflects upon the subtle harmonies and the affinities of number and of magnitude. Its dormant qualities flower when it gradually solves a few "small simple postulates and a system of interesting theorems expanding into infinite and unexpected uses and applications." Mathematics is more than an organized knowledge of things, facts and events,

in their true relation and coordination, their antecedents and consequences. "No number of facts or aphorisms learned by heart makes a man a thinker, or does him much intellectual service." The elementary truths of number and space lie quite outside the region of contingency or controversy, and in consequence furnish ample and reliable material for the acquisition of habits of accurate thinking and deductive reasoning, so indispensable to success in all the pursuits of life, whether in material or spiritual activity. "Take a geometrical axiom, an elementary truth concerning the properties of space—two straight lines cannot enclose a space—or in arithmetic the elementary truth concerning the properties of number—to multiply by two numbers successively, is to multiply by their product, and we observe that the moment we state them, we perceive the necessary truth, there is no room for debate or difference of opinion; to understand either statement is to accept it. And so with all other fundamental axioms of mathematics. Whatever particular facts prove ultimately to be contained in these general or universal truths must be true. As far as we can be certain of anything we are certain of these."

St. Augustine appreciated the value of mathematical certitude, and made applications of it in his search for a knowledge of God and the soul. He employed the same plan of logical reasoning, and placed comparatively little worth on knowledge received through sense perception, unless followed up by careful investigation and reinforced by the intellect. "Wherefore it seems to me, that one could more easily sail on land, than learn geometry by means of the senses. If the knowledge of God and geometry were on a par, I would rejoice as much in knowing geometry, as I presume I shall rejoice once I know God. Now, however, in comparison with the knowledge of God, I so greatly contend, knowledge of geometry, that sometimes it seems to me, that if I knew Him as I see He can be known, I should quite forget my knowledge of geometry. Since indeed, in the presence of His love, the thought of geometry scarcely comes to my mind, and I am led to acknowledge that in its own field as far as the earth differs from heaven, so far do those true and definite truths differ from the intelligible majesty of God."

How often have not purely mathematical symbols, because of their symplicity, been used to point out abstract supernatural

truths to untrained minds. St. Augustine finds in the equilateral triangle and the square, the symbol of justice. The circle is the typical figure for eternity, since it has neither a beginning nor an end. The superimposed triangles, the star of the House of David, led the Jews of the Old Law while the chosen people lived in the light of God's blessings, and today it is still their symbol of hope of the Messiah yet to come. Cardinal Nikalous of Cusa compared the center of the circle to God as the efficient cause of the universe, indivisible and simple. "God's perfect simplicity does not consist merely in His indivisibility, but primarily, in the simultaneous plenitude of His positive perfections of being." The locus of all points equally distant from a common point is a circle, or in accordance with the genetic definition, a circle is a figure formed by a point moving in its plane at a constant distance from another point in the same plane.

Among the characteristics of a cultured intellectuality I name the defense of ideality, and the maintenance of spirituality. It is objected, ideality and spirituality can have no place in mathematics. Cold, brutal, inelastic mathematics dealing with its rules, its angles and its circles. Stop for but a moment and consider, the very essence of the science, and it will be seen at once that it is from its very nature abstract. Indeed its very abstractness is usually of a higher order than the abstractedness of the logician. Thus we find imagination entering into the citadel of truth, and certitude, and claiming the right of intimate companionship with the multiplication table and the propositions of Euclid. The whole study of geometry is an imaginative study. The lines with which the mathematician deals are not imperfect lines, drawn by the straightest pen, not the finest gossamer web, woven by Queen Mab's tiniest spider, but ideal lines which have length without breadth, and which therefore, can exist only in the cultured imagination of the highly trained mathematician. The sharpest blade produced in Damascus and sweeping, with all but infinite swiftness, into the most faintly resisting substance, would leave an angle too crude to be dealt with by the mathematician; so he retires to the quiet confines of his cultured mathematical mind, and there, from the fabric from which the ordinary mortal weaves his dreams, he constructs his angles whose sides are perfect, and

places them as his offering on the altar of absolute mathematical truth. No eye has ever seen or ever will see a circle or a square which complies with the definition of the circle or the square of the mathematician. The thing defined exists only in the mathematician's imagination, and every proposition in geometry involves the exercise of that faculty.

Most of us are forced to admit that the workings of our imaginations are bounded by the three dimensions of space in which we live and move—length, breadth and thickness. Out of them we model our mental conceptions, but not so the mathematician. He forms an imaginative conception of space, of four or five, or "*n* dimensions" and draws up laws which would govern his imaginative universe with absolute precision if its walls of mystic fiction were metamorphosed into the imperfect materials of fact. The idealist in those abstruse branches of science which deal with the problems of the ultimate construction of matter, and of the laws governing the forces which act upon it, arms himself not alone with his instruments of precision, his rules and his formulae, but finds them powerless without the aid of his mathematical imagination.

The astronomer climbs to his observatory and with his space-defying lenses peers into the hidden mysteries of a thousand worlds. With uncanny certainty he predicts phenomena to generations as yet unborn with a precision that confidently invites the closest scrutiny of his fellows, he weighs and measures the heavenly bodies, maps their courses, and calculates their velocity as they sweep through space. I am convinced that the lense of a Copernicus or Galileo would be worthless if hand and eye had not been directed by a truly cultured imaginative mind.

And so one might go through all the sciences, and find that those intellects that are productive of most valuable results in the realm of original research are strange combinations of the real and the unreal, the practical and the imaginative, each quality of mind, essential for the proper fulfillment of the other.

"For the dreamer lives on forever,
But the toiler dies in a day."

I have endeavored to show the close relationship existing between mathematics, by its very nature, and culture in its truest sense. That culture demands love of truth, that mathematics

depends essentially upon the attainment of truth, that true culture transcends the material things of life, that mathematics in its abstractions refuses to be bound by physical laws. It remains only to show that in its practical application mathematics remains a faithful servant to that idealism which points the way to higher and to better things, and brings man to a better knowledge and understanding of his Creator.

It may be said without fear of contradiction that there is no calling to which men have devoted themselves that has not received aid from applied mathematics. If then, mathematics has as its ultimate end the attainment of truth, it cannot but have been a faithful servant in the universality of its workings.

The architect, whether he follow any of the three great schools of the world, the Greek, Romanesque or Gothic, applies his mathematical rules to the straight line, the curve, and the angle, and behold, the Parthenon, the Coliseum, Rheims and St. Peter's spring into being. The Chaldeans, the greatest mathematicians of ancient times, in the beautiful tile construction and mosaics of the Alhambra, and the Mosque of Omar at Jerusalem, have given the modern world something better than tales of debauchery and bloodshed by which they were otherwise to be remembered. The mystery of the sphinx, the square beauty of the pyramids, and the scientific construction of their obelisks, have given to the world a sweeter dream of the Egypt of old, than the story of the power of the Pharaohs, or the corruption of Cleopatra. The roads of ancient Rome, over which rattled the chariot of all-conquering Caesar, as he led his troops to triumph, remain as a monument to his mathematical genius, though

> "Imperial Caesar—dead and turned to clay,
> May serve to stop a hole and keep the wind away."

If our museums are filled with collections of the sculptured arts of the ancients, and if the walls of the galleries and cathedrals of the world are rich with the colors of the masters of the brush, if bridges span our streams, if levies hold back the angry floods, if men speed across land and water, and if they have conquered the air, if messages of sorrow or rejoicing, of loss or gain, travel lightning-like through space, if contented labor pours its incense from a million forges to the heaven, then, indeed, has the science of mathematics justified its creation, for none of these successes could have been had not 2 plus 2 made

4, had not truth proved its might against doubt and ignorance and error.

Wherever ignorance is conquered, wherever error is overcome, man is brought into more healthful relationship with his fellow-man, and into closer communion with his Creator. St. Augustine has said, "That is the true perfection of a man, to find out his own imperfections." No science which unduly exalted a creature would truly serve him. Mathematics pointing always to absolute truth as its ultimate goal, points also to the limitations of all human endeavor, to the frailty of finite intelligence, to man's inability to travel, in the workings of any of his faculties beyond the narrow confines set for him by the Master who brought him out of nothingness.

Thus mathematics by developing the higher intellectual faculties of the individual, making him the unselfish servant of his brother, and bringing him into closer knowledge of his Maker, fulfills the requirement of a cultural science. If pride closed the gates of heaven against the vanity of Lucifer and man in intellectual pride attempts to follow him, he must first leave by the roadside that faithful companion in dull drab raiment, the burden of whose song as he trudges along is ever "2 plus 2 makes 4," "Truth is mighty and must prevail."

DISCUSSION

Grouping Pupils According to Ability. The mortality in ninth year mathematics is relatively large in the average high school. Some of the reasons for this state of affairs are (1) the course of study is too traditional; (2) the work is not made cumulative with respect to the work that has been done in the seventh and eighth grades; (3) many pupils have not learned how "to study" mathematics before they reach the ninth grade; (4) many pupils have been promoted to the ninth year because they are either unable or unwilling to do the work in the eighth year satisfactorily.

Statistics show that pupils in the ninth year range in mental age from ten and one-half years to sixteen and even as high as eighteen years. West Technical High School is no exception in this respect. We have in the past been attempting to compel every student to do the same kind of work, in the same way, under the same kind of instructions. Such a procedure, however, is folly; yea, it is dangerous; furthermore, it is unfair to the majority of pupils. On the one hand, it does not give the pupil of lower mentality a chance at all; his time is practically wasted. On the other hand, it gives the pupil of super-normal ability no incentive, but encourages him to become an incessant loafer. It is the business of a school to give every pupil an inspiration and an incentive to work toward a "top-notch" goal, an ideal absolutely impossible under the old system.

For two semesters we have placed the pupils in the 9-B and 9-A classes, at West Technical High School, into three groups; viz., (1) group (a), group (b), and group (c); or the super-normal group, the normal group, and the sub-normal group. The classification of pupils for these groups is based upon three main facts; viz., (1) their rank as indicated by certain intelligence test; (2) their rank as indicated by certain diagnostic tests; and (3) upon the teacher's estimate of the pupil's ability.

At least two or three classes of the same grade are scheduled the same period; so that the various groups may have their work simultaneously, which permits an easy shift from one group to another, according to the kind of work the pupil is doing.

The pupil is told frankly, and in all sincerity, the group to which he belongs, and that he may be placed into a stronger group at any time that he shows sufficient ability; and vice versa.

In this classification, group (a) is given not only the minimum requirement of the course of study as laid out by the Board of Education, but also the work recommended to be omitted because of lack of time, and much additional work planned by the teacher. In some cases pupils work out special mathematical "projects." These pupils will eventually become the leaders in the various walks of life.

Group (b) is given the minimum requirement of the course of study. In some cases some additional work can be done.

Group (c) is given a more differentiated course. The pupils in this group take the simplest types of problems in the regular course, and then fill in with more concrete material of mensurational geometry and arithmetic, as well as giving more time to drill upon the four fundamental operations with integral and fractional numbers.

Pupils, in this group, who do their work conscientiously, are given a passing credit, and their mark on the permanent record card indicates a special credit. At the end of the ninth year, these pupils are advised to discontinue their mathematics and pursue other lines of work, which function more satisfactorily than mathematics. If, however, such a pupil insists that he must continue his mathematics, he is required to take the regular 9-A course and do the work satisfactorily, before he attempts the course in demonstrative geometry.

Suffice it to say here that the pupils in the tenth year are classified into two groups; viz., group (a) and (b). After the tenth year the courses in mathematics are entirely elective, except for certain specialized courses, and no further grouping is necessary.

This method of grouping pupils according to ability has proved very successful and satisfactory at West Technical High School. The following are some of the benefits derived from it:

1. Pupils are happier because they can progress more rapidly together in their own group.

2. The more brilliant pupils are not learning lazy habits; it gives them a goal for which to work.

3. Pupils who are inclined to work more slowly do not rush over unlearned fundamentals in an attempt to keep up.

4. The curriculum can be adapted to individual differences, and hence the work can be made to function.

5. The method of teaching can be varied to suit. It is obvious that the same method is not successful with both the accelerated and the slow pupil. Many a pupil has been taught to be a loafer under the old system.

6. Teachers are much happier in their work, because they feel that they can have at least some standard of achievement.

7. The percentage of failures has been reduced from more than 30% to less than 10%. As the system is perfected from time to time, the percentage of failures may be reduced to a negligible quantity.

D. W. Werremeyer.

West Technical High School, Cleveland, Ohio.

Tennis Ball Geometry. What better thing than that a "dead" tennis ball, of the vintage of 1921, should by "dead"—section teach the true inwardness of the sphere to his young master!

> I hold it truth, with him who sings
> To one clear harp in divers tones,
> That balls may rise on stepping-stones
> Of their dead selves to higher things.
> —With apologies to Tennyson.

For example:

1. To describe a circumference (horizon circle) on a sphere with a given pole (zenith) and a given polar chord.

A tennis ball and a compass.

2. To find the radius of the circle described above. Mark three points on the circumference. With the compass construct on a separate sheet a congruent triangle (three sides respectively equal). Find the circumcenter of this triangle. Draw the circumscribed circle. Check. Cut this circle out. The hole should fit *closely* the circle on the tennis ball.

3. To find the diameter of a sphere (tennis ball).

Take a section through the axis of the circle. Make a cut-out on cardboard. The tennis ball should just pass through. If not, repeat.

4. Describe parallel circles and hence construct a zone.

Use the same pole.

5. Construct a line.

Use the great circle cut out of 3. Etc.

6. Each face angle of a central polyhedral angle is measured by the subtended side of its spherical polygon.

Lay out on cardboard, that will fold, consecutively four angles of quite distinct size as *e. g.*, 30°, 60°, 90° and 120°. With common vertex as a center and radius equal to that of the ball describe circle. Cut out, fold and paste. Mark and cut out corresponding spherical polygon from ball. Cut out an opposite polygon. Cut out a third congruent polygon.

7. Each dihedral angle of a central polyhedral angle has the same measure as the corresponding spherical angle of its spherical polygon. cf. 6. Etc.

8. Opposite spherical polygons are symmetric. "Symmetry vs. Congruence." Use the polygons of 6. Etc.

9. Two isosceles symmetric spherical triangles are congruent. After the cut outs are fitted the proof is but an incident. Etc.

10. What are polar triangles? Use the great circle cut out and apply the definition. Etc.

HOWARD F. HART.

Montclair, N. J.

NEWS AND NOTES

THE Illinois Section of the Mathematical Association of America is arranging for a meeting in connection with the meeting of the Illinois State Academy of Science at Rockford, Ill., on April 28th and 29th.

The program consists of the following:

Friday, 2 P. M.—

1. Constructive Methods in Geometry. Prof. Emch, University of Illinois.
2. Some Aspects of Correlation Theory. Mr. Mensenkamp, Freeport High School.
3. Romance in Science—An Experimental Course Offered by a Mathematics Department. Prof. Bessie Miller, Rockford College.
4. Consistency in Grading Mathematics Papers. Prof. E. J. Moulton, Northwestern University.

Friday Night—

5. An illustrated lecture on "Cosmogony," by Prof. MacMillan of the University of Chicago before a joint meeting of the Illinois Section and the State Academy of Science.

Saturday, 9 A. M.—

6. The National Committee Report on College Entrance Requirements in Mathematics (published in THE MATHEMATICS TEACHER for May, 1921). Discussion led by Prof. Wahlin, University of Illinois; Dr. Kinney, Crane Junior College, and Prof. Parsons, DeKalb State Teachers' College.
7. How Many and What Mathematics Courses Should Be Offered to College Freshmen. Discussion led by Prof. Scott, Illinois College; Prof. Ginnings, Macomb State Teachers' College.

C. E. Comstock, E. J. Moulton, and E. B. Lytle, Program Committee.

THE former students of the Department of Mathematics of Bryn Mawr College announce a mathematical meeting to be held in Taylor Hall on Tuesday, April 18th, in honor of Prof. Charlotte Angas Scott, D.Sc., on the completion of her thirty-seventh year as head of the Department of Mathematics in Bryn Mawr

College. The program includes: Address of welcome, President M. Carey Thomas, Ph.D., LL.D., L.H.D.; introductory address, Miss Marion Reilly, A.B., 1901; speaker, Prof. Alfred North Whitehead, Sc.D., F.R.S., professor of applied mathematics in the Imperial College of Science, South Kensington, subject, "Relativity and Gravitation. Group Tensors and Their Application to the Formulation of Physical Laws."

The honorary committee consists of twenty distinguished mathematicians.

THE *High School Research Bulletin* (March, 1922) of the Los Angeles city schools contains a series of special articles on subjects of the high school curriculum by Dr. J. F. Bobbitt of the University of Chicago. They are written as a basis of further work by the Course of Study Committees. Prof. Bobbitt's "general assumptions and principles" for the Los Angeles curriculum in mathematics are:

1. The mathematics to be included in the general training should be determined by what men and women actually need in their general affairs outside of their several callings; and by the common mathematical element of all vocations.

2. The major thing needed is not ability to solve difficult mathematical problems; it is rather ability and disposition to think accurately and quantitatively in one's affairs. The latter frequently involves mathematical operations as incidental matters—never as the fundamental ones.

3. The way to learn to think quantitatively is mainly to think quantitatively in those various fields where quantitative thought is possible and desirable.

4. The ability to do quantitative thinking is to be developed in youth under conditions as nearly like those in which it is to function in adulthood, as practicable.

5. The ability to think quantitatively is a general need. It should therefore be a portion or aspect of the required training.

6. While the mathematical operations are not the main things, yet it is indispensable that one perform the needed ones with certainty and skill.

7. Outside of their vocations, the citizens of Los Angeles do not use algebra, demonstrational geometry, or trigonometry.

8. Outside of their vocations, the only mathematics content really needed by the men and women of the city is applied arithmetic.

9. Even in their vocations only a small percentage of the citizens of Los Angeles use algebra or trigonometry; and practically none use demonstrational geometry.

10. The mathematics needed for one's vocation should be determined strictly with a view to that vocation. It should then be administered only to those who enter that vocation; and it should be very thorough, especially along the applied lines involved in that vocation.

11. As fields of intellectual play, neither algebra nor demonstrational geometry lay foundations or centers of systems of ideas and thought generally needed throughout life.

12. As matters of pure general discipline, the city cannot afford to

administer algebra and geometry purely on faith: the specific disciplinary values should be made clear; and it should be demonstrated that they are or can be attained.

13. The value of applied mathematics, intensive and thorough, as a discipline for producing power to think, to assemble and organize facts, etc., has been amply and indisputably demonstrated.

14. The content of the economic and civic studies needs to be developed so as to include the necessary large amount of applied mathematics.

15. The mathematical element of the science studies, particularly general science and the biological sciences, needs much further development.

16. The needed mastery of the world of number is to be attained mainly through using number—not by studying abstractions about number.

17. The needed mastery of the world of form and space-relations is to be attained mainly by using forms and by constructing forms that are to be used. Studies about forms need be only brief and incidental.

Miss Fannie S. Mitchell was elected President of the Mathematics Section of the North Carolina State Teachers' Association which met at the Greenville Normal School, February 3rd and 4th. Mr. Raleigh Schorling of The Lincoln School of Teachers' College gave a series of talks on "The Mathematics of Grades Seven, Eight and Nine" to joint meetings of the Association and the Normal School.

Prof. David Eugene Smith is traveling in Europe during his sabbatical leave from Teachers College. He will return for the winter session (October, 1922) at Teachers' College.

The Detroit Mathematics Club offered the following program during 1921-22:

October 28th, Prof. Louis C. Karpinski, University of Michigan, "Mathematics and Life."

January 12th, Miss Orpha E. Worden, Detroit Teachers' College, "Teaching of High School Mathematics"; Miss Mabel C. Woodward, Detroit Teachers' College, "The Mathematics Situation in the Intermediate Schools."

March 16th, Address by Prof. A. E. Lyman, Ypsilanti State Normal College.

April 4th, Prof. Cassius J. Keyser, Columbia University, "The Mathematical Obligations of Philosophy and Education."

The program of the Mathematics Section of the fifty-seventh meeting of the Michigan Schoolmasters' Club, March 30 and 31, 1922, consisted of the following:

Chairman, Jane L. Matteson, Michigan State Normal College.

Secretary, John Craig, Muskegon.

1. "How to Encourage the Formation of Right Habits in First Year Algebra," Miss Alice M. Woessner, Ann Arbor.

2. "The Value of the Long Recitation Period in High School Mathematics," Principal George W. Murdock, S. W. High School, Detroit.

3. "First Year Algebra for Pupils of Varying Capacities," Miss Selma Lindell, Flint.

4. "The Segregation of Classes According to Mental Tests," Mr. Isaac M. De Voe, Highland Park.

5. General Discussion. Opened by Professor R. A. Wells, Michigan State Normal College.

Friday afternoon, March 31:

6. Report of the Chicago meeting of National Council of Mathematics Teachers, Miss Orpha E. Worden, Teachers' College, Detroit.

7. "Specific Causes of Failure in University Work in Mathematics," Professor M. F. Johnson, University of Michigan.

8. Discussion of Professor Rouse's Paper of last year on "Causes of Failure of Students of the Engineering College," Miss Gladys Snyder, Muskegon.

9. "The Ohio Ruling," Mr. Harold Blair, Western Normal School.

10. General Discussion. Opened by Dean C. B. Williams, Kalamazoo College.

THE Educational Association of Western Pennsylvania met at Pittsburgh, March 27, 1922.

At the Mathematics Section, the following program was given:

"The Course in Mathematics as Recommended by the National Committee—Will This Fit the Student for His Future Work?" Prof. Paul Webber, Department of Mathematics, University of Pittsburgh.

Discussion.

"The Relation of College Preparatory to Vocational Mathematics," Prof. Glenn James, Department of Mathematics, Carnegie Institute of Technology.

Discussion.

Chairman, Miss Jane Matthews, Peabody High School.

Secretary, Mr. Redenbaugh, Peabody High School.

THE Philadelphia Teachers' Association conducted a series of departmental conferences Saturday, April 1, 1922. The program of the Mathematics Section was arranged by Philadelphia Section of the Association of Teachers of Mathematics of the Middle States and Maryland, and consisted of:

1. Election of officers.

2. Address, "Tests of Mathematical Abilities," Prof. E. L. Thorndike, Institute of Educational Research, Teachers' College, Columbia University.

Officers: President, Albert H. Wilson, Haverford College; Vice President, C. Burton Walsh, Friends' Central School; Secretary-Treasurer, Alice M. Holbrook, Philadelphia High School for Girls.

Executive Committee: K. Eleanor Cooper, Germantown High School; Samuel K. Brecht, Central High School.

Discussion.

THE CHICAGO MEETING OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The business meeting was called to order at 10 o'clock A. M. by Dr. J. H. Minnick, President of the Council.

The Secretary read the minutes of the last meeting, which were approved.

The Secretary then reported on the mailing list of MATHEMATICS TEACHER, giving the total number of copies mailed out for February, 1922, as 2,681, and giving the number in each of the states. The report of the finances of the journal indicate that the Council is solvent, and gives promise of continued and increased usefulness.

There was considerable discussion of the need for greatly increasing the number of subscribers of MATHEMATICS TEACHER, and of methods of effectively bringing the journal and its possibilities to the attention of mathematics teachers throughout the country.

Vigorous propaganda on behalf of MATHEMATICS TEACHER is to be carried on by representatives of the National Council at all meetings of state and local organizations of teachers of mathematics during the coming year.

Mr. J. R. Clark, editor of MATHEMATICS TEACHER, reported that the publisher has finally succeeded in getting up a schedule for the regular appearance of the journal; and that in the future there probably will be few delays. There is much available material, and enthusiastic co-operation in securing good material in some centers. More attention should be given in the future, by local representatives, to the matter of providing complete reports of meetings of teachers of mathematics, and to transmitting copies of papers presented at these meetings for publication. Various speakers stressed the importance of enabling representatives to say that paper will appear later in the pages of MATHEMATICS TEACHER.

On motion, the Chairman was authorized to appoint a publicity committee to advertise the National Council and MATHEMATICS TEACHER.

The impossibility of any conflict of interest between the Central Association of Science and Mathematics Teacher and the

National Council was briefly commented upon by Alfred Davis, President of the Central Association. President Minnick emphasized the importance of local organizations.

On motion the question of revision of the section in the constitution referring to membership was referred to the Executive Committee.

On motion, the question of increasing the number of the Executive Committee was referred to the latter body.

On motion, the question of becoming a department of the N. E. A. was referred to the Executive Committee.

On motion by Mr. Austin, the Chairman was authorized to appoint a committee to confer with the new President of the Department of Superintendence of the National Education Association with a view to re-establish relations with the Department of Superintendence.

The following members of the Executive Committee were present at its meeting: Dr. Minnick, E. H. Taylor, Miss Worden, Mr. C. M. Austin, Miss Gugle, Mr. Foberg.

The Executive Committee took the following action on questions referred to it:

It was moved that all reference to collective membership in the constitution be stricken out. Carried.

It was moved by Miss Gugle that the Secretary be instructed to publish regularly in MATHEMATICS TEACHER, a list of the state representatives of the National Council. Carried.

On motion, it was decided not to become a department of the National Education Association.

At the afternoon meeting of the National Council, the action taken by the Executive Committee on these various questions was reported and approved.

President Minnick announced the following committees:

Nominating Committee: Mr. C. M. Austin, Oak Park, Ill.; Miss L. Carter, St. Joseph, Mo.; Mr. C. B. Walsh, Philadelphia, Pa.

Publicity: John R. Clark, Raleigh Schorling, C. B. Walsh.

The remainder of the afternoon was taken up with the program.

Dr. Kinney of the Crane Junior College, Chicago, read a paper on "The Function Concept in Secondary School Mathematics."

Prof. J. W. Young reported on "Some Aspects of the Work of the National Committee on Mathematical Requirements."

Prof. Slaught occupied the place assigned originally to Prof. Hedrick, and talked in interesting fashion of the various methods that have been proposed for reorganizing secondary school mathematics, and of the possibilities for economizing time in mathematics education. Mr. Breslich and Mr. W. W. Hart discussed aspects of the questions raised by Prof. Slaught.

Mr. Alfred Davis spoke on "Some Problems in Secondary School Mathematics."

Shortly after the adjournment of the afternoon session, the members of the National Council sat down to a dinner, arranged jointly by the two mathematics clubs of Chicago and its vicinity. Our enjoyment of the banquet was greatly increased by the seating arrangements thoughtfully planned in advance by the Committee with Mrs. Devereaux as chairman.

At the close of the dinner, Prof. Slaught presided genially and efficiently as toastmaster.

His first duty was to receive the report of the Nominating Committee, made by Mr. C. M. Austin. The following names were put in nomination, and the report unanimously approved: For President, Prof. J. H. Minnick, of Philadelphia; for Vice President, Miss Eula Weeks of St. Louis; for Executive Committee, Gertrude Allen, Oakland, Cal., and Mr. Rankin, of North Carolina.

Mr. W. D. Reeve spoke on the theme, "General Mathematics."

Prof. G. W. Myers spoke on the theme, "Reactionary vs. Progressive in Mathematics Teaching."

Mr. Schorling discussed, in happy fashion, the question: "Is Mathematics Teaching Responding to Modern Demands in Secondary Education?"

Dr. Minnick spoke inspiringly of the possibilities that lie before the National Council in the direction of influencing the teaching of mathematics.

At the close of the long day of meetings and discussions, all agreed that the time had been profitably spent. Much credit was given the very efficient committees that had in charge the arrangements for the various meetings.

J. A. Foberg, *Secretary-Treasurer.*

THE following named persons registered at the meetings of the National Council:

Gertrude E. Allen, University High School, Oakland, Cal.; Gladys Crisman, Blue Island High School, Blue Island, Ill.; E. H. Taylor, Eastern State Teachers' College, Charleston, Ill.; E. R. Breslich, School of Education, University of Chicago; Fannie J. Brown, Crane Technical High School, Chicago, Ill.; Mary Devereux, Austin High School, Chicago, Ill.; William W. Gorsline, Crane Junior College, Chicago, Ill.; Anna E. Hill, Crane Technical High School, Chicago, Ill.; Ruth E. Hopewell, Hyde Park High School, Chicago, Ill.; Frances I. Hubler, Tilden Technical High School, Chicago, Ill.; Ethel Jaynes, Lane Technical High School, Chicago, Ill.; Stella M. Johnson, Tilden Technical High School, Chicago, Ill.; Lillian E. Kurtz, Fenger High School, Chicago, Ill.; F. R. Liddil, Crane Technical High School, Chicago, Ill.; Myrta E. Mercill, Bowen High School, Chicago, Ill.; Wilson L. Miser, Armour Institute of Technology, Chicago, Ill.; G. W. Myers, College of Education, University of Chicago; George W. Oldfather, Crane Technical High School, Chicago, Ill.; C. J. Palmer, Armour Institute of Technology, Chicago, Ill.; Cora B. Peerstone, Crane High School, Chicago, Ill.; C. A. Petterson, Carl Schuz High School, Chicago, Ill.; J. C. Piety, Crane Technical High School, Chicago, Ill.; Miss N. M. Quinn, Tilden Technical High School, Chicago, Ill.; Beulah I. Shoesmith, Hyde Park High School, Chicago, Ill.; Mrs. Ruth O. Witt, Tilden Technical High School, Chicago, Ill.; G. A. Sorrich, Elmhurst Junior College, Elmhurst, Ill.; Clara L. Hughes, Evanston Township High School, Evanston, Ill.; M. Estelle Nash, Evanston Township High School, Evanston, Ill.; M. J. Newell, Evanston High School, Evanston, Ill.; George G. Taylor, High School, Highland Park, Ill.; Christine H. MacMartin, High School, Highland Park, Ill.; Dorothy Hinman, High School, Highland Park, Ill.; Margaret E. Mills, Deerfield Shields High School, Highland Park, Ill.; Florence J. Morgan, Deerfield Shields High School, Highland Park, Ill.; Earl L. Thompson, Junior College and Township H. S., Joliet, Ill.; George A. Harper, New Trier, Kenilworth, Ill.; M. Hildebrandt, Proviso Township High School, Maywood, Ill.; Edwin W. Schreiber, Proviso Township High School, Maywood, Ill.; Dorothy Wagner, Proviso Township High School, Maywood, Ill.; M. W. Coultrap,

Northwestern College, Naperville, Ill.; C. M. Austin, Oak Park, Ill.; Mrs. Elsie Parker Johnson, Oak Park High School, Oak Park, Ill.; H. W. Chandler, Rockford High School, Rockford, Ill.; Miss Georgiana H. Fischer, St. Charles High School, St. Charles, Ill.

Bonnie L. Shoop, Streator Township High School, Streator, Ill.; Ernest B. Lytle, University of Illinois, Urbana, Ill.; Bess Dady, Waukegan Township Junior School, Waukegan, Ill., Margaret M. Dady, Waukegan Township Junior H. S., Waukegan, Ill.; Grace I. Smith, Waukegan Junior High School, Waukegan, Ill.; R. H. Shanks, Culver Military Academy, Culver, Ind.; S. R. Wells, East Chicago, Ind.; W. G. Gingery, Shortridge High School, Indianapolis, Ind.; Kate Wentz, Emmerich Manual High School, Indianapolis, Ind.; John E. Dotterer, Manchester College, North Manchester, Ind.; Ira S. Condit, Iowa State Teachers' College, Cedar Falls, Iowa; A. A. Ellsworth, Amasa, Mich.; Sadie M. Alley, Northwestern High School, Detroit, Mich.; Harry M. Keal, Cass Technical High School, Detroit, Mich.; J. V. McNally, Northwestern High School, Detroit, Mich.; Mabel C. Woodward, Western High and Teachers' College, Detroit, Mich.; Orpha E. Worden, Detroit Teachers' College, Detroit, Mich.; Isaac M. DeVoe, Highland Park High School, Highland Park, Mich.; W. D. Reeve, University High School, Minneapolis, Minn.; Leolian Carter, Central High and Junior College, St. Joseph, Mo.; Alfred Davis, Soldan High School, St. Louis, Mo.; Albert H. Huntington, Cleveland High School, St. Louis, Mo.; J. W. Young, Dartmouth College, Hanover, N. H.; J. R. Clark, Lincoln School, New York City; Raleigh Schorling, The Lincoln School of Teachers' College, New York City; Arthur F. M. Petersilge, East High School, Cleveland, Ohio; Marie Gugle, Assistant Superintendent of Schools, Columbus, Ohio; E. C. Vermillion, State Department of Education, Columbus, Ohio; J. A. Foberg, Director of Mathematics in Pennsylvania, Harrisburg, Pa.; J. H. Minnick, University of Pennsylvania, Philadelphia, Pa.; C. B. Walsh, Friends' School, Philadelphia, Pa.; Ethel L. Budd, Appleton High School, Appleton, Wis.; Ethel S. Carter, Appleton High School, Appleton, Wis.; Esther M. Raaflaub, Deerfield High School, Deerfield, Wis.; Dora E. Kearney, High School, Finnimore, Wis.; Mrs. Sarah E. Decker, Fond du Lac High School, Fond du Lac, Wis.; Elizabeth Churchill, Kem-

per Hall, Kenosha, Wis.; Mary Louise Williams, Kenosha High School, Kenosha, Wis.; L. K. Adkins, State Normal School, La Crosse, Wis.; Walter W. Hart, University of Wisconsin, Madison, Wis.; Ruth M. Allen, Riverside High School, Milwaukee, Wis.; Henry Ericson, Washington High School, Milwaukee, Wis.; Elder L. Swanson, Riverside High School, Milwaukee, Wis.; Paul W. Waterman, Milwaukee Country Day School, Milwaukee, Wis.; Mrs. Rose Wagner Bruins, Racine High School, Racine, Wis.; Ida E. Howe, Racine High School, Racine, Wis.; Anna L. Neitzel, Racine High School, Racine, Wis.; Mary A. Potter, Racine High School, Racine, Wis.; Dorothy A. Rott, Racine High School, Racine, Wis.; Florence L. Stockly, Racine High School, Racine, Wis.; Mrs. Jean Cowles, Stoughton High School, Stoughton, Wis.; O. H. Bigelow, Whitewater, Wis.

RESEARCH DEPARTMENT*

Purpose of a Research Department. Many teachers realize that the results of their teaching could be markedly improved by the solution of a number of problems. For example, we do not know definitely the best method for making our pupils more accurate in computation. It would be easy to make a long list of problems that challenge careful thought. THE MATHEMATICS TEACHER may possibly serve as a means of mobilizing thought on these problems and as an organ for placing the valid claims of mathematics with reference to the debated questions before the public. Some problems require that the data be assembled from many sources, reflecting a wide selection of pupils, teachers, courses and the like. It seems that THE MATHEMATICS TEACHER can be particularly helpful in attacking problems that require extensive cooperation. It is fascinating to speculate what forward strides the teaching of mathematics might be able to take if three mathematics teachers in each State gave from two to four hours each month to the solution of some one well-defined problem.

*Beginning with this month, The Mathematics Teacher will include a section devoted to Research Problems. Communications should be directed to Mr. Raleigh Schorling, who is in charge of this department.